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Exercise Set 1: Multiple Integrals

Analysis 3 — Chapter 1

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Exercise 1

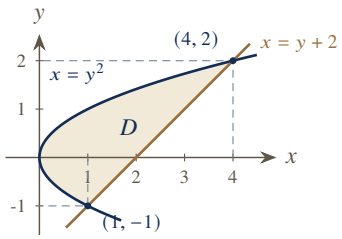
1. Compute the limit
$$L = \lim_{n\to+\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{n}{(2n+i)(n^2+j^2)}.$$

2. Let $f : [0,1]\times[0,1] \to \mathbb{R}$ be defined by
$$f(x,y) = \begin{cases} 1-x-y, & \text{if } y \neq x, \\ 3, & \text{if } y = x. \end{cases}$$
Show that f is integrable on $[0,1]^2$ and compute $\displaystyle\iint_{[0,1]^2} f(x,y)\,dx\,dy$.

3. Let $g : [0,1]\times[0,1] \to \mathbb{R}$ be defined by
$$g(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{(x^2+y^2)^3}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$
Compute $I_1 = \displaystyle\int_0^1 \left(\int_0^1 g(x,y)\,dy\right) dx$ and $I_2 = \displaystyle\int_0^1 \left(\int_0^1 g(x,y)\,dx\right) dy$. Is g integrable on $[0,1]^2$? Justify.

Exercise 2

1. Let $D \subset \mathbb{R}^2$ be the domain bounded by $x = y^2$ and $x = y + 2$ represented below.



For a continuous function f on D, express $\displaystyle\iint_D f(x,y)\,dx\,dy$ in two different ways, then compute $\displaystyle\iint_D y\,dx\,dy$.

2. Interchange the order of integration and compute
$$J = \int_0^1 \left(\int_{y/2}^{y} e^{x^2}\,dx\right) dy + \int_1^2 \left(\int_{y/2}^{1} e^{x^2}\,dx\right) dy.$$

3. Sketch the domain $D_1 = \left\{(x,y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 0,\ 2y \leq x^2+y^2 \leq 2\right\}$ and compute $\displaystyle\iint_{D_1} x\,e^y\,dx\,dy$.

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4. Compute, in two different ways, the integral

$$J = \iint_{[0,1]\times[0,1]} \frac{x}{(1+x^2)(1+xy)}\,dx\,dy.$$

Deduce the value of $I = \displaystyle\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx$.

Exercise 3

Compute $\displaystyle\iint_D f(x,y)\,dx\,dy$ in each of the following cases:

(a) $f(x,y) = \cos\left(\dfrac{x-y}{x+y}\right)$, D is bounded by the lines $x=0$, $y=0$, $x+y=1$, and $x+y=2$.

(b) $f(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}$, $D = \left\{(x,y)\in\mathbb{R}^2 : x\geq 0,\ 1\leq x^2+y^2\leq 2y\right\}$.

(c) $f(x,y) = (y^2-x^2)^{xy}\,(x^2+y^2)$, $D = \left\{(x,y)\in\mathbb{R}^2 : 0<x<y,\ 1<xy<2,\ y^2-x^2<1\right\}$.

(d) $f(x,y) = (y^2-x^2)\cos(xy)$, $D = \left\{(x,y)\in\mathbb{R}^2 : 0\leq x\leq y,\ x+y\leq 4,\ xy\geq 1\right\}$.

Exercise 4

1. Compute the area of the planar domain D bounded by the curves $y=ax^2$, $y=bx^2$, $xy=c$, and $xy=d$ $(0<a<b, 0<c<d, x>0, y>0)$.

2. Compute the mass M, the average density Q_M, and the center of gravity (x_G, y_G) of the plate $D = \left\{(x,y)\in\mathbb{R}^2 : y\geq 0,\ 1\leq x^2+y^2\leq 4\right\}$ with density $\rho(x,y) = y\sqrt{x^2+y^2}$.

3. Let P be a homogeneous plate of mass M obtained by removing the disk $x^2+y^2<R^2$ from the square $[-2R,2R]^2$ $(R>0)$. Express the moments of inertia I_O, I_x, and I_y of P in terms of M and R.

4. Compute the moment of inertia I_Δ of a homogeneous triangular plate $D = \left\{(x,y)\in\mathbb{R}^2 : x\geq 0,\ y\geq 0,\ x+y\leq a\right\}$ $(a>0$, density $\rho_0>0)$ with respect to the line $\Delta : x+y=a$.

Exercise 5

Compute $\displaystyle\iiint_D f(x,y,z)\,dx\,dy\,dz$ in each of the following cases:

(a) $f(x,y,z) = \dfrac{1}{(1+x+y+z)^3}$, $D = \left\{(x,y,z)\in\mathbb{R}^3 : x\geq 0,\ y\geq 0,\ z\geq 0,\ x+y+z\leq 1\right\}$.

(b) $f(x,y,z) = y$, D is bounded by the paraboloid $y=x^2+z^2$ and the plane $y=4$.

(c) $f(x,y,z) = 2z+x$, $D = \left\{(x,y,z)\in\mathbb{R}^3 : z\geq 0,\ y\geq x^2-1,\ y+z\leq 1\right\}$.

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Exercise 6

Compute $\iiint_D f(x,y,z)\,dx\,dy\,dz$ in each of the following cases:

(a) $f(x,y,z) = \dfrac{z}{\sqrt{x^2+y^2+z^2}}$, $\quad D = \left\{(x,y,z) \in \mathbb{R}^3 : 1 \le x^2+y^2+z^2 \le 4,\ z \ge \sqrt{x^2+y^2}\right\}$.

(b) $f(x,y,z) = z\,e^{x^2+y^2}$, $\quad D = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le 1,\ z \ge 0\right\}$.

(c) $f(x,y,z) = \sqrt{x^2+y^2+(z-1)^2}$, $\quad D = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le 4\right\}$.

(d) $f(x,y,z) = xyz + \dfrac{x+y+z}{xyz}$, $\quad D = \left\{(x,y,z) \in \mathbb{R}^3 : 1 \le xy \le 4,\ 1 \le yz \le 9,\ 1 \le xz \le 4\right\}$.

Exercise 7

Determine the volumes of the following solids:

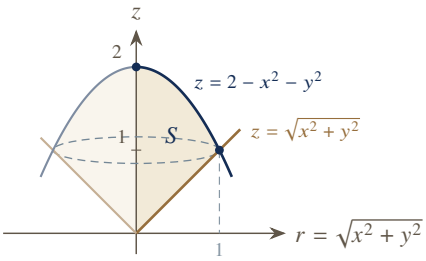
(a) $V_1 = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le R^2,\ x^2+y^2-Rx \le 0\right\}$, $\quad R > 0$.

(b) $V_2 = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+2y^2+6z^2+2xy+2xz \le 1\right\}$.

(c) $V_3 = \left\{(x,y,z) \in \mathbb{R}^3 : \left(\dfrac{x}{a}\right)^{2/3} + \left(\dfrac{y}{b}\right)^{2/3} + \left(\dfrac{z}{c}\right)^{2/3} \le 1\right\}$, $\quad a,b,c > 0$.

Exercise 8

Let $S \subset \mathbb{R}^3$ be the solid bounded by the cone $z = \sqrt{x^2+y^2}$ and the paraboloid $z = 2 - x^2 - y^2$ (see figure below).



1. Compute the volume of S.
2. Assume S has density $\rho(x,y,z) = \rho_0\sqrt{x^2+y^2}$ ($\rho_0 > 0$). Compute the mass M and the center of inertia (x_G, y_G, z_G) of S.
3. For the same density, compute the moments of inertia I_{xOy}, I_{yOz}, I_{xOz} and I_{Ox}, I_{Oy}, I_{Oz} of S.