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Exercise Set 2: Vector Calculus

Analysis 3 — Chapter 2

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Exercise 1

Throughout this exercise, $\mathbb{R}^3$ is equipped with its canonical orthonormal basis $(\vec{i}, \vec{j}, \vec{k})$, and we set $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$ and $r = \|\vec{r}\| = \sqrt{x^2 + y^2 + z^2}$ on $U = \mathbb{R}^3 \setminus \{(0,0,0)\}$.

1. Let $\varphi(x,y,z) = x^2 y - y z^3$ and $\vec{A}(x,y,z) = yz\,\vec{i} - xz\,\vec{j} + x^2 y\,\vec{k}$ on $\mathbb{R}^3$. Compute $\nabla\varphi$, $\Delta\varphi$, $\operatorname{div}\vec{A}$, $\operatorname{rot}\vec{A}$, $\operatorname{div}(\operatorname{rot}\vec{A})$, and $\operatorname{rot}(\nabla\varphi)$.

2. Let $g \in C^2(]0,+\infty[,\mathbb{R})$ and define $u(x,y,z) = g(r)$ on U.

   (a) Prove that $\nabla u(x,y,z) = \dfrac{g'(r)}{r}\,\vec{r}$ and $\Delta u(x,y,z) = g''(r) + \dfrac{2}{r}\,g'(r)$ on U.

   (b) Determine all functions $g \in C^2(]0,+\infty[,\mathbb{R})$ such that $\Delta u = 0$ on U.

3. For $\alpha \in \mathbb{R}$, define $\vec{F}_\alpha(x,y,z) = r^\alpha\,\vec{r}$ on U.

   (a) Compute $\operatorname{div}\vec{F}_\alpha$ and $\operatorname{rot}\vec{F}_\alpha$ on U, and determine the value of α for which $\operatorname{div}\vec{F}_\alpha = 0$ on U.

   (b) Show that $\vec{F}_\alpha$ is conservative on U for every $\alpha \in \mathbb{R}$, and determine all its scalar potentials on U (distinguishing the case $\alpha = -2$).

Exercise 2

1. Fix $a > 0$ and consider the two planar curves

$$\gamma_1(t) = \big(a\cos^3 t,\ a\sin^3 t\big),\quad t \in \left[0, \frac{\pi}{2}\right], \qquad \gamma_2(u) = \Big(au^3,\ a(1-u^2)^{3/2}\Big),\quad u \in [0,1].$$

   (a) Show that the restriction of γ_2 to $]0,1[$ is a C^1 reparametrization of the restriction of γ_1 to $\left]0, \frac{\pi}{2}\right[$, and specify whether it preserves or reverses orientation.

   (b) Determine the regular points of γ_1 on $\left[0, \frac{\pi}{2}\right]$, find the unit tangent vector $\vec{T}(t)$ and unit normal vector $\vec{N}(t)$ at every regular point, and compute the arc length $\ell(\gamma_1)$.

2. Consider the space curve $\gamma : \mathbb{R} \to \mathbb{R}^3$ defined by

$$\gamma(t) = \begin{cases} \big((a^2 - 2a)t,\ bt + t^2,\ 0\big), & \text{if } t < 0, \\ \big(\sin(at),\ \sin\big((3b - b^2)t\big),\ t^3\big), & \text{if } t \geq 0, \end{cases}$$

   where $(a,b) \in \mathbb{R}^2$. Determine all pairs $(a,b) \in \mathbb{R}^2$ such that γ is of class C^1 on $\mathbb{R}$ and regular at every $t \in \mathbb{R}$.

3. Determine all $k \in \mathbb{R}$ such that the arc length of $\gamma(t) = \left(2t,\ kt^2,\ \dfrac{1}{3}t^3\right)$ on $[0,T]$ equals $2T + \dfrac{1}{3}T^3$ for every $T > 0$.

4. Express the arc length element $ds = \|\gamma'(t)\|\,dt$ of a C^1 space curve given in cylindrical coordinates by $x(t) = r(t)\cos\theta(t)$, $y(t) = r(t)\sin\theta(t)$, $z(t) = z(t)$ in terms of r, θ, z, and their derivatives. Compute the length of the conical spiral $r(t) = e^t$, $\theta(t) = \sqrt{2}\,t$, $z(t) = e^t$ for $t \in [0, \ln 3]$.

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Exercise 3

1. Consider the differential 1-form on $\mathbb{R}^3$ defined by

$$\omega = \left(2xye^z + \alpha z^2\right) dx + \left(x^2e^z + 3y^2z\right) dy + \left(\beta x^2ye^z + 2xz + y^3\right) dz, \qquad (\alpha, \beta) \in \mathbb{R}^2.$$

   Determine (α, β) so that ω is exact on $\mathbb{R}^3$, and find all functions $f \in C^1(\mathbb{R}^3, \mathbb{R})$ satisfying $df = \omega$.

2. Let $\varphi, \psi \in C^1(\mathbb{R}, \mathbb{R})$ satisfy $\psi(0) = 1$, and consider the vector field on $\mathbb{R}^3$ given by

$$\vec{V}(x, y, z) = yz\,\varphi(x)\,\vec{i} + 2z\,\psi(x)\,\vec{j} + \Big(y\big(\varphi(x) + \psi(x)\big) + 2z\Big)\vec{k}.$$

   (a) Compute $\operatorname{div}\vec{V}$ and $\operatorname{rot}\vec{V}$ in terms of φ, ψ, and their derivatives.

   (b) Determine φ and ψ such that $\vec{V}$ is conservative on $\mathbb{R}^3$, and find all scalar potentials of $\vec{V}$ on $\mathbb{R}^3$.

3. On the open half-plane $U = \{(x, y) \in \mathbb{R}^2 : y > 0\}$, consider $\vec{F}(x, y) = 2xy\,\vec{i} + (y^2 - 2x^2)\,\vec{j}$.

   (a) Verify that $\vec{F}$ is not conservative on U.

   (b) Find the unique function $\mu \in C^1(]0, +\infty[, \mathbb{R}^*)$ with $\mu(1) = 1$ such that $\mu(y)\,\vec{F}(x, y)$ is conservative on U, and determine all scalar potentials of $\mu(y)\,\vec{F}(x, y)$ on U.

Exercise 4

1. Let Γ be the helical arc parameterized by $\gamma(t) = \left(2\cos t,\ 2\sin t,\ \sqrt{5}\,t\right)$ for $t \in [0, \pi]$.

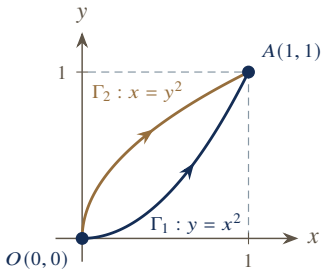
   (a) Compute the mass $M = \displaystyle\int_\Gamma \rho(x, y, z)\, ds$ of the wire Γ with linear density $\rho(x, y, z) = z + x^2y$, and give the value of $\displaystyle\int_{\Gamma^-} \rho(x, y, z)\, ds$, where Γ^- denotes Γ with the opposite orientation.

   (b) Prove that

$$\frac{3\pi}{\sqrt{9 + 25\pi^4}} \le \int_\Gamma \frac{ds}{\sqrt{5 + x^2 + y^2 + z^4}} \le \pi.$$

2. Fix $a > 0$ and let C be the cardioid parameterized by $\gamma(t) = \left(a(1 + \cos t)\cos t,\ a(1 + \cos t)\sin t\right)$ for $t \in [0, 2\pi]$. Compute $\displaystyle\oint_C |y|\, ds$.

3. For $\lambda \in \mathbb{R}$, let $\vec{F}_\lambda(x, y) = (2xy + y^2)\,\vec{i} + (\lambda x^2 + 2xy)\,\vec{j}$ on $\mathbb{R}^2$. Consider the two oriented parabolic arcs $\Gamma_1 : y = x^2$ and $\Gamma_2 : x = y^2$ from $O(0, 0)$ to $A(1, 1)$ shown below.



   Compute $W_1(\lambda) = \displaystyle\int_{\Gamma_1} \vec{F}_\lambda \cdot d\vec{s}$ and $W_2(\lambda) = \displaystyle\int_{\Gamma_2} \vec{F}_\lambda \cdot d\vec{s}$. Determine the value of λ for which $W_1(\lambda) = W_2(\lambda)$, show that $\vec{F}_\lambda$ is then conservative on $\mathbb{R}^2$, and deduce $\displaystyle\int_\Gamma \vec{F}_\lambda \cdot d\vec{s}$ for every piecewise C^1 curve Γ from O to A.

4. Let Γ be the oriented planar curve parameterized by $\gamma(t) = \left(\cos t + \sin(2t),\ \sin t + \sin^2 t\right)$ for $t \in [0, \pi]$. Compute

$$I = \int_\Gamma \frac{1 - x^2 - 2xy + y^2}{(1 + x^2 + y^2)^2}\, dx + \frac{1 + x^2 - 2xy - y^2}{(1 + x^2 + y^2)^2}\, dy.$$

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Exercise 5

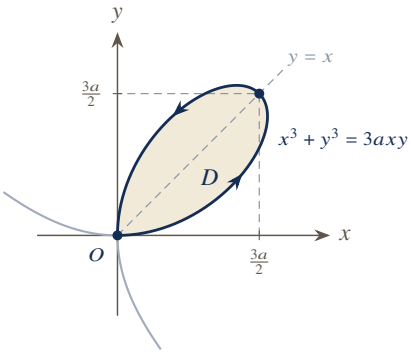
On the punctured plane $\Omega = \mathbb{R}^2 \setminus \{(0,0)\}$, consider the differential 1-form

$$\omega = \left(2xy - \frac{2y}{x^2+4y^2}\right)dx + \left(x^2 + \frac{2x}{x^2+4y^2}\right)dy.$$

1. Show that ω is closed on Ω. Why does Poincaré's theorem not apply to ω on Ω?
2. Let C_1 be the ellipse $x^2 + 4y^2 = 4$ oriented counterclockwise. Compute $\displaystyle\oint_{C_1} \omega$. Is ω exact on Ω? Justify.
3. Let C_2 be the circle $(x-3)^2 + y^2 = 4$ oriented counterclockwise. Compute $\displaystyle\oint_{C_2} \omega$.
4. Show that the restriction of ω to the open right half-plane $U_+ = \{(x,y) \in \mathbb{R}^2 : x > 0\}$ is exact, find all its primitives on U_+, and compute $\displaystyle\int_\Gamma \omega$ along any C^1 curve $\Gamma \subset U_+$ joining $A(2,-1)$ to $B(2,1)$.

Exercise 6

1. Let $D = \{(x,y) \in \mathbb{R}^2 : y \geq 0,\ 1 \leq x^2 + y^2 \leq 3\}$, and let Γ^+ denote its positively oriented boundary. Compute
$$I = \oint_{\Gamma^+} \left(e^{x^2}\sin x - y^3 + 2y\right)dx + \left(x^3 + 4x + \ln(2+\cos y)\right)dy.$$
2. Fix $a, b > 0$ and let Γ be the upper half-ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $y \geq 0$, oriented from $A(a,0)$ to $B(-a,0)$. Compute
$$J = \int_\Gamma \left(e^x \cos y - 3y\right)dx + \left(x - e^x \sin y\right)dy.$$
3. Fix $a > 0$ and let $D \subset \mathbb{R}^2$ be the domain bounded by the loop of the folium of Descartes $x^3 + y^3 = 3axy$ $(x \geq 0,\ y \geq 0)$ represented below.



   (a) Setting $y = tx$ $(t \geq 0)$, parameterize the loop for $(x,y) \neq (0,0)$ and specify the orientation induced as t increases.

   (b) Compute the area $\mathcal{A}(D)$.

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Exercise 7

1. Consider the parameterized surface S_1 given by

$$\vec{r}(u,v) = u\cos v\,\vec{i} + u\sin v\,\vec{j} + u\,\vec{k}, \qquad (u,v) \in \mathbb{R}\times[0,2\pi].$$

Find a Cartesian equation satisfied by the points of S_1, compute the fundamental cross product $\dfrac{\partial\vec{r}}{\partial u}\times\dfrac{\partial\vec{r}}{\partial v}$, and determine all pairs $(u,v)\in\mathbb{R}\times[0,2\pi]$ at which S_1 fails to be smooth.

2. Consider the parameterized surface S_2 defined by

$$\vec{\Phi}(u,v) = (u\cos v - \sin v)\,\vec{i} + (u\sin v + \cos v)\,\vec{j} + u\,\vec{k}, \qquad (u,v)\in\mathbb{R}\times[0,2\pi[.$$

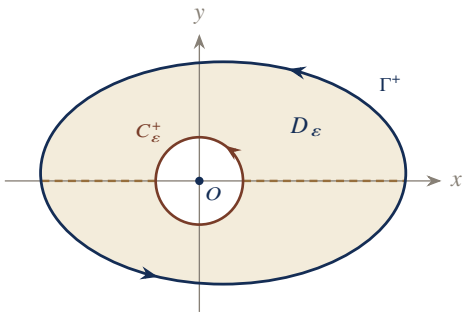
   (a) Verify that every point (x,y,z) of S_2 lies on the quadric $x^2+y^2-z^2=1$, compute $\vec{N}(u,v) = \dfrac{\partial\vec{\Phi}}{\partial u}\times\dfrac{\partial\vec{\Phi}}{\partial v}$, and prove that S_2 is smooth everywhere on $\mathbb{R}\times[0,2\pi[$.

   (b) Determine the Cartesian equation of the tangent plane to S_2 at the point corresponding to $(u_0,v_0) = (2,0)$.

   (c) Determine all pairs $(u,v)\in\mathbb{R}\times[0,2\pi[$ at which the tangent plane to S_2 is parallel to the plane $x+y-z=0$, and write the Cartesian equations of the corresponding tangent planes.

Exercise 8

Let $D\subset\mathbb{R}^2$ be a bounded domain containing $(0,0)$ in its interior, bounded by a positively oriented piecewise C^1 simple closed curve Γ^+. Fix $\varepsilon>0$ small enough so that the closed disk $x^2+y^2\le\varepsilon^2$ lies inside D, set $D_\varepsilon = \{(x,y)\in D : x^2+y^2\ge\varepsilon^2\}$, and let C_ε^+ denote the circle $x^2+y^2=\varepsilon^2$ oriented counterclockwise (see figure below).



1. Let $P,Q\in C^1\big(\mathbb{R}^2\setminus\{(0,0)\}\big)$. Prove that

$$\oint_{\Gamma^+}(P\,dx+Q\,dy) - \oint_{C_\varepsilon^+}(P\,dx+Q\,dy) = \iint_{D_\varepsilon}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy.$$

2. Consider $\omega_0 = \dfrac{-y\,dx+x\,dy}{x^2+y^2}$. Explain why the Green–Riemann formula cannot be applied directly to ω_0 on D, then deduce from Question 1 the value of $\displaystyle\oint_{\Gamma^+}\omega_0$.

3. Compute

$$K_1 = \oint_{\Gamma_1^+}\left(\frac{x-2y}{x^2+y^2} - y + \sin(x^2)\right)dx + \left(\frac{2x+y}{x^2+y^2} + x + e^{y^2}\right)dy,$$

where Γ_1^+ is the counterclockwise boundary of the rectangle $R_1=[-1,3]\times[-1,2]$.

4. Compute the integral of the same differential form over the counterclockwise boundary Γ_2^+ of the rectangle $R_2=[1,3]\times[-1,2]$.