Analysis 3
Exercise Set 1

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Multiple Integrals — Detailed Correction

Analysis 3 — Detailed Correction – Exercise Set 1

Focus on the Way of Thinking, Explicit Substitutions, and Step-by-Step Course Execution

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How to Use This Correction

Each solution follows a strict four-stage analytical structure designed to train exam intuition: **Method** (identifies the exact course technique) ⟶ **Why we do this** (explains the mental train of thought and algebraic clues) ⟶ **Execution** (complete line-by-line derivation with zero skipped steps and explicit **Sub** callouts) ⟶ **Pitfall / Sanity Check** (where students lose points and how to verify your answer).

Difficulty: ★☆☆☆☆ Direct ★★☆☆☆ Standard ★★★☆☆ Exam level ★★★★☆ Tricky ★★★★★ Synthesis

Exercise 1

1. Compute the limit $L = \lim_{n\to+\infty} \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{n}{(2n+i)(n^2+j^2)}$.

★★☆☆☆

Method *Double Riemann Sum Interpretation and Separable-Variable Factorization on a Rectangle*

Why we do this (Way of Thinking):
A double sum indexed by $1 \le i \le n$ and $1 \le j \le n$ represents a double Riemann sum over the unit square $R = [0,1]\times[0,1]$ provided we can isolate the area element $\Delta A = \Delta x\,\Delta y = \frac{1}{n}\cdot\frac{1}{n} = \frac{1}{n^2}$ and express the remaining factor solely in terms of the ratios $\frac{i}{n}$ and $\frac{j}{n}$. Factoring n out of $(2n+i)$ and n^2 out of (n^2+j^2) produces n^3 in the denominator; combined with the n in the numerator, $\frac{n}{n^3} = \frac{1}{n^2}$ supplies the exact area factor ΔA.

Execution: Step 1 — Algebraic reformulation as a Riemann sum: For every $n \ge 1$ and $1 \le i, j \le n$, factor n from the first bracket and n^2 from the second bracket in the denominator:

$$\begin{aligned}\frac{n}{(2n+i)(n^2+j^2)} &= \frac{n}{n\left(2+\frac{i}{n}\right)\cdot n^2\left(1+\frac{j^2}{n^2}\right)}\\ &= \frac{1}{\left(2+\frac{i}{n}\right)\left(1+\left(\frac{j}{n}\right)^2\right)}\cdot\frac{1}{n^2}.\end{aligned}$$

Partition the unit square $R = [0,1]\times[0,1]$ into n^2 uniform subrectangles $R_{ij} = \left[\frac{i-1}{n}, \frac{i}{n}\right]\times\left[\frac{j-1}{n}, \frac{j}{n}\right]$ of widths $\Delta x = \frac{1}{n}$ and $\Delta y = \frac{1}{n}$, area $\Delta A = \Delta x\,\Delta y = \frac{1}{n^2}$, and upper-right sample points $(x_i^*, y_j^*) = \left(\frac{i}{n}, \frac{j}{n}\right) \in R_{ij}$. Define $h : [0,1]\times[0,1] \to \mathbb{R}$ by

$$h(x,y) = \frac{1}{(2+x)(1+y^2)}.$$

Then the finite double sum equals the double Riemann sum

$$S_n = \sum_{i=1}^{n}\sum_{j=1}^{n} h(x_i^*, y_j^*)\,\Delta A.$$

Step 2 — Passage to the double integral and separable evaluation: Because $2+x \ge 2 > 0$ and $1+y^2 \ge 1 > 0$ on $R = [0,1]\times[0,1]$, the rational function h is continuous on the closed rectangle R, hence Riemann-integrable on R. Passing to the limit as $n \to +\infty$ yields

$$L = \lim_{n\to+\infty} S_n = \iint_{[0,1]\times[0,1]} \frac{1}{(2+x)(1+y^2)}\,dx\,dy.$$

Since $h(x,y) = g_1(x)\,g_2(y)$ has separable variables with $g_1(x) = \frac{1}{2+x}$ and $g_2(y) = \frac{1}{1+y^2}$ over the rectangle $[0,1]\times[0,1]$, the double integral factors into the product of two single-variable integrals:

$$\begin{aligned} L &= \left(\int_0^1 \frac{1}{2+x}\,dx\right)\left(\int_0^1 \frac{1}{1+y^2}\,dy\right) \\ &= \Big[\ln(2+x)\Big]_{x=0}^{x=1} \cdot \Big[\arctan y\Big]_{y=0}^{y=1} \\ &= \big(\ln 3 - \ln 2\big)\big(\arctan 1 - \arctan 0\big) \\ &= \ln\left(\frac{3}{2}\right)\cdot\left(\frac{\pi}{4} - 0\right). \end{aligned}$$

$$\boxed{L = \frac{\pi}{4}\ln\left(\frac{3}{2}\right)}$$

> ► **Common Pitfall:**
> In two dimensions, the Riemann area element on $[0,1]^2$ is $\Delta A = \frac{1}{n^2}$, not $\frac{1}{n}$. If after factoring out powers of n the remaining multiplier were $\frac{1}{n}$ (giving $n\cdot\frac{1}{n^2}$), the sum would diverge to $+\infty$; if it were $\frac{1}{n^3}$, the limit would be 0.

> ✓ **Sanity Check:**
> On $[0,1]^2$, the integrand satisfies $\frac{1}{3\cdot 2} = \frac{1}{6} \le \frac{1}{(2+x)(1+y^2)} \le \frac{1}{2\cdot 1} = \frac{1}{2}$. Integrating over $\text{Area}(R) = 1$ requires $\frac{1}{6} \le L \le \frac{1}{2}$. Numerically, $\frac{\pi}{4}\ln(1.5) \approx 0.7854 \times 0.4055 \approx 0.3185 \in [0.1667, 0.5]$.

2. Let $f : [0,1]\times[0,1] \to \mathbb{R}$ be defined by $f(x,y) = 1 - x - y$ if $y \neq x$, and $f(x,y) = 3$ if $y = x$. Show that f is integrable on $[0,1]^2$ and compute $\iint_{[0,1]^2} f(x,y)\,dx\,dy$. ★★★☆☆

Method *Darboux Integrability Criterion via Upper and Lower Sums on Uniform Partitions*

> ***Why we do this (Way of Thinking):***
> The function f coincides everywhere with the continuous plane $f_0(x,y) = 1 - x - y$ except along the diagonal segment $y = x$, where its value jumps to 3. Because f is discontinuous along $y = x$, we cannot invoke continuity directly on $[0,1]^2$; instead, we apply the Darboux integrability criterion on the uniform $n \times n$ grid P_n. The key geometric observation is that among the n^2 closed subsquares R_{ij}, almost all subsquares stay completely away from the line $y = x$, while the number of closed subsquares touching $y = x$ grows only linearly like $O(n)$. Multiplying $O(n)$ jump subsquares by the subsquare area $\frac{1}{n^2}$ forces their contribution to $S(f,P_n) - s(f,P_n)$ to vanish at rate $O\left(\frac{1}{n}\right)$.

Execution: Step 1 — Boundedness of f on $R = [0,1]^2$: For $(x,y) \in [0,1]^2$ with $y \neq x$, we have $0 \le x+y \le 2$, so $-1 \le 1 - x - y \le 1$. On the diagonal $y = x$, $f(x,x) = 3$. Consequently,

$$-1 \le f(x,y) \le 3 \qquad \forall (x,y) \in [0,1]^2,$$

which confirms that f is bounded on R.

Step 2 — Partition P_n and classification of closed subsquares: For each integer $n \ge 2$, let P_n be the uniform partition dividing $[0,1]^2$ into n^2 closed subsquares

$$R_{ij} = \left[\frac{i-1}{n}, \frac{i}{n}\right] \times \left[\frac{j-1}{n}, \frac{j}{n}\right], \qquad 1 \le i, j \le n, \qquad \Delta A_{ij} = \frac{1}{n^2}.$$

Set $m_{ij} = \inf_{(x,y)\in R_{ij}} f(x,y)$ and $M_{ij} = \sup_{(x,y)\in R_{ij}} f(x,y)$. We examine when the closed square R_{ij} intersects the diagonal $\Delta = \{(x,y) \in [0,1]^2 : y = x\}$:

- **Case A ($|i-j|\geq 2$, off-diagonal subsquares):** If $j\geq i+2$, then for every $(x,y)\in R_{ij}$, $y\geq \frac{j-1}{n}\geq \frac{i+1}{n}>\frac{i}{n}\geq x$; similarly, if $i\geq j+2$, then $x\geq \frac{i-1}{n}\geq \frac{j+1}{n}>\frac{j}{n}\geq y$. Thus $R_{ij}\cap\Delta=\emptyset$, so $f(x,y)=1-x-y$ everywhere on R_{ij}. Since $1-x-y$ decreases in both x and y:

$$M_{ij}=1-\frac{i-1}{n}-\frac{j-1}{n},\qquad m_{ij}=1-\frac{i}{n}-\frac{j}{n}\quad\Longrightarrow\quad M_{ij}-m_{ij}=\frac{1}{n}+\frac{1}{n}=\frac{2}{n}.$$

- **Case B ($|i-j|\leq 1$, subsquares meeting Δ):** These consist of the n diagonal squares $i=j$, the $n-1$ super-diagonal squares $j=i+1$ (which touch Δ at the single corner $\left(\frac{i}{n},\frac{i}{n}\right)$), and the $n-1$ sub-diagonal squares $j=i-1$ (which touch Δ at the single corner $\left(\frac{i-1}{n},\frac{i-1}{n}\right)$), giving $n+2(n-1)=3n-2$ subsquares in total. Since $R_{ij}\cap\Delta\neq\emptyset$, $M_{ij}=3$. For the infimum, every $(x,y)\in R_{ij}$ satisfies $f(x,y)\geq 1-x-y\geq 1-\frac{i}{n}-\frac{j}{n}$ (noting $3>1\geq 1-\frac{i}{n}-\frac{j}{n}$), and choosing interior points $(x_k,y_k)\in R_{ij}$ with $y_k\neq x_k$ converging to $\left(\frac{i}{n},\frac{j}{n}\right)$ gives $f(x_k,y_k)=1-x_k-y_k\to 1-\frac{i}{n}-\frac{j}{n}$. Hence

$$m_{ij}=1-\frac{i}{n}-\frac{j}{n}\geq -1\quad\Longrightarrow\quad M_{ij}-m_{ij}\leq 3-(-1)=4.$$

Step 3 — Darboux difference estimate and integrability: Summing $(M_{ij}-m_{ij})\,\Delta A_{ij}$ over all n^2 subsquares (comprising $n^2-(3n-2)$ squares in Case A and $3n-2$ squares in Case B):

$$\begin{aligned}0\leq S(f,P_n)-s(f,P_n)&=\sum_{|i-j|\geq 2}(M_{ij}-m_{ij})\frac{1}{n^2}+\sum_{|i-j|\leq 1}(M_{ij}-m_{ij})\frac{1}{n^2}\\&\leq \left(n^2-3n+2\right)\cdot\frac{2}{n}\cdot\frac{1}{n^2}+(3n-2)\cdot 4\cdot\frac{1}{n^2}\\&\leq n^2\cdot\frac{2}{n^3}+3n\cdot\frac{4}{n^2}=\frac{2}{n}+\frac{12}{n}=\frac{14}{n}.\end{aligned}$$

Since $\lim_{n\to+\infty}\frac{14}{n}=0$, for every $\varepsilon>0$ there exists an integer $n>\frac{14}{\varepsilon}$ such that $S(f,P_n)-s(f,P_n)<\varepsilon$. By the Darboux integrability criterion, f is Riemann-integrable on $[0,1]^2$.

Step 4 — Exact evaluation of the double integral: From Step 2, on *every* subsquare R_{ij} $(1\leq i,j\leq n)$ we obtained $m_{ij}=1-\frac{i}{n}-\frac{j}{n}$, which coincides identically with the local infimum of the continuous function $f_0(x,y)=1-x-y$ on R_{ij}. Because $s(f,P_n)\leq\iint_{[0,1]^2}f(x,y)\,dx\,dy\leq S(f,P_n)$ and $S(f,P_n)-s(f,P_n)\to 0$, the double integral equals $\lim_{n\to+\infty}s(f,P_n)=\iint_{[0,1]^2}f_0(x,y)\,dx\,dy$. Using $\sum_{k=1}^{n}1=n$ and $\sum_{k=1}^{n}k=\frac{n(n+1)}{2}$, we evaluate $s(f,P_n)$ directly (and verify by Fubini's theorem on f_0):

$$\begin{aligned}s(f,P_n)&=\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(1-\frac{i}{n}-\frac{j}{n}\right)=\frac{1}{n^2}\left(n^2-\frac{n}{n}\sum_{i=1}^{n}i-\frac{n}{n}\sum_{j=1}^{n}j\right)\\&=\frac{1}{n^2}\left(n^2-\frac{n(n+1)}{2}-\frac{n(n+1)}{2}\right)=\frac{n^2-n^2-n}{n^2}=-\frac{1}{n}\xrightarrow[n\to+\infty]{}0.\end{aligned}$$

Equivalently, applying Fubini's theorem to the continuous function $f_0(x,y)=1-x-y$ on $[0,1]^2$:

$$\begin{aligned}\iint_{[0,1]^2}f(x,y)\,dx\,dy&=\int_0^1\left(\int_0^1(1-x-y)\,dy\right)dx=\int_0^1\left[(1-x)y-\frac{y^2}{2}\right]_{y=0}^{y=1}dx\\&=\int_0^1\left(\frac{1}{2}-x\right)dx=\left[\frac{x}{2}-\frac{x^2}{2}\right]_0^1=0.\end{aligned}$$

f is Riemann-integrable on $[0,1]^2$, and $\iint_{[0,1]^2}f(x,y)\,dx\,dy=0$

> ► **Common Pitfall:**
> Because Darboux subrectangles $R_{ij} = \left[\frac{i-1}{n}, \frac{i}{n}\right] \times \left[\frac{j-1}{n}, \frac{j}{n}\right]$ are closed, the off-diagonal squares $R_{i,i+1}$ and $R_{i,i-1}$ still touch the line $y = x$ at a single shared corner vertex, so $M_{ij} = 3$ on $3n - 2$ subsquares rather than only on the n diagonal squares R_{ii}. Accounting for $|i - j| \leq 1$ makes the upper-sum bound mathematically airtight.

> ✓ **Sanity Check:**
> Under the center-point reflection $(x, y) \mapsto (1 - x, 1 - y)$ of $[0, 1]^2$, any off-diagonal point $y \neq x$ maps to $1 - y \neq 1 - x$, and $f_0(1 - x, 1 - y) = 1 - (1 - x) - (1 - y) = -(1 - x - y) = -f_0(x, y)$, confirming by odd symmetry about $\left(\frac{1}{2}, \frac{1}{2}\right)$ that $\iint_{[0,1]^2} f_0(x, y)\, dx\, dy = 0$.

3. Let $g : [0, 1] \times [0, 1] \to \mathbb{R}$ be defined by $g(x, y) = \dfrac{xy(x^2 - y^2)}{(x^2 + y^2)^3}$ if $(x, y) \neq (0, 0)$, and $g(0, 0) = 0$. Compute $I_1 = \displaystyle\int_0^1 \left(\int_0^1 g(x, y)\, dy\right) dx$ and $I_2 = \displaystyle\int_0^1 \left(\int_0^1 g(x, y)\, dx\right) dy$. Is g integrable on $[0, 1]^2$? Justify. ★★★★☆

Method ***Iterated Rational-Substitution Integration, Boundedness Test, and Contrapositive of Fubini's Theorem***

> ***Why we do this (Way of Thinking):***
> When $x > 0$ is held fixed, the denominator $(x^2 + y^2)^3$ never vanishes on $y \in [0, 1]$, and the factor $y\, dy$ in the numerator is proportional to the differential of $u = x^2 + y^2$. Rewriting the remaining factor $x^2 - y^2 = 2x^2 - (x^2 + y^2) = 2x^2 - u$ splits the inner rational integrand into two immediate negative powers of u. Comparing I_1 and I_2, together with probing the growth of $g(x, y)$ along a ray $y = kx$ approaching $(0, 0)$, then decides whether g can be integrable on $[0, 1]^2$.

Execution: Step 1 — Computation of $I_1 = \displaystyle\int_0^1 \left(\int_0^1 g(x, y)\, dy\right) dx$:

- At $x = 0$: $g(0, 0) = 0$ by definition, and for every $y \in (0, 1]$, $g(0, y) = \frac{0 \cdot y(0 - y^2)}{(0 + y^2)^3} = 0$. Hence $\int_0^1 g(0, y)\, dy = 0$.
- For fixed $x \in (0, 1]$: the function $y \mapsto g(x, y)$ is continuous on $[0, 1]$ since $x^2 + y^2 \geq x^2 > 0$.

Sub **For fixed $x \in (0, 1]$, let $u = x^2 + y^2 \implies \boldsymbol{du = 2y\, dy} \implies y\, dy = \frac{1}{2}\, du$, with $y^2 = u - x^2 \implies x^2 - y^2 = 2x^2 - u$. Bounds: $y = 0 \implies u = x^2$; $y = 1 \implies u = x^2 + 1$.**

Evaluating the inner integral for $x \in (0, 1]$:

$$\begin{aligned}\int_0^1 \frac{xy(x^2 - y^2)}{(x^2 + y^2)^3}\, dy &= \frac{x}{2} \int_{x^2}^{x^2+1} \frac{2x^2 - u}{u^3}\, du = \frac{x}{2} \int_{x^2}^{x^2+1} \left(2x^2 u^{-3} - u^{-2}\right) du \\ &= \frac{x}{2}\left[2x^2\left(\frac{u^{-2}}{-2}\right) - \left(\frac{u^{-1}}{-1}\right)\right]_{u=x^2}^{u=x^2+1} = \frac{x}{2}\left[-\frac{x^2}{u^2} + \frac{1}{u}\right]_{u=x^2}^{u=x^2+1} = \frac{x}{2}\left[\frac{u - x^2}{u^2}\right]_{u=x^2}^{u=x^2+1} \\ &= \frac{x}{2}\left(\frac{(x^2 + 1) - x^2}{(x^2 + 1)^2} - \frac{x^2 - x^2}{x^4}\right) = \frac{x}{2(1 + x^2)^2}.\end{aligned}$$

Observe that this formula also gives $\frac{0}{2(1+0)^2} = 0$ at $x = 0$, so $\int_0^1 g(x, y)\, dy = \frac{x}{2(1+x^2)^2}$ for all $x \in [0, 1]$. We now integrate with respect to x over $[0, 1]$:

Sub **Let $w = 1 + x^2 \implies \boldsymbol{dw = 2x\, dx} \implies x\, dx = \frac{1}{2}\, dw$. Bounds: $x = 0 \implies w = 1$; $x = 1 \implies w = 2$.**

$$I_1 = \int_0^1 \frac{x}{2(1 + x^2)^2}\, dx = \frac{1}{4} \int_1^2 w^{-2}\, dw = \frac{1}{4}\left[-\frac{1}{w}\right]_{w=1}^{w=2} = \frac{1}{4}\left(-\frac{1}{2} + 1\right) = \frac{1}{8}.$$

Step 2 — Computation of $I_2 = \displaystyle\int_0^1 \left(\int_0^1 g(x,y)\,dx \right) dy$: At $y = 0$, $g(x,0) = 0$ for all $x \in [0,1]$, so $\int_0^1 g(x,0)\,dx = 0$. For fixed $y \in (0,1]$, $x \mapsto g(x,y)$ is continuous on $[0,1]$.

Sub **For fixed $y \in (0,1]$, let $v = x^2 + y^2 \implies \boldsymbol{dv = 2x\,dx} \implies x\,dx = \frac{1}{2}\,dv$, with $x^2 - y^2 = v - 2y^2$. Bounds: $x = 0 \implies v = y^2$; $x = 1 \implies v = y^2 + 1$.**

$$\int_0^1 \frac{xy(x^2-y^2)}{(x^2+y^2)^3}\,dx = \frac{y}{2}\int_{y^2}^{y^2+1} \frac{v-2y^2}{v^3}\,dv = \frac{y}{2}\int_{y^2}^{y^2+1} \left(v^{-2} - 2y^2 v^{-3}\right) dv$$

$$= \frac{y}{2}\left[-\frac{1}{v} + \frac{y^2}{v^2}\right]_{v=y^2}^{v=y^2+1} = \frac{y}{2}\left[\frac{y^2 - v}{v^2}\right]_{v=y^2}^{v=y^2+1} = \frac{y}{2}\left(\frac{-1}{(y^2+1)^2} - 0\right) = -\frac{y}{2(1+y^2)^2},$$

which again holds at $y = 0$. Integrating over $y \in [0,1]$ via the same substitution $w = 1 + y^2$ ($\boldsymbol{dw = 2y\,dy}$, $w \in [1,2]$):

$$I_2 = -\int_0^1 \frac{y}{2(1+y^2)^2}\,dy = -\frac{1}{4}\int_1^2 w^{-2}\,dw = -\frac{1}{8}.$$

Step 3 — Non-integrability of g on $[0,1]^2$: The function g is **not integrable** on $[0,1]^2$, for two decisive reasons:

(i) **Failure of boundedness (Riemann definition):** A function must be bounded on $[0,1]^2$ to be Riemann-integrable. Evaluating g along the ray $y = \frac{1}{2}x$ for $x \in (0,1]$ (so $\left(x, \frac{x}{2}\right) \in [0,1]^2$) yields

$$g\left(x, \frac{x}{2}\right) = \frac{x \cdot \frac{x}{2}\left(x^2 - \frac{x^2}{4}\right)}{\left(x^2 + \frac{x^2}{4}\right)^3} = \frac{\frac{3}{8}x^4}{\left(\frac{5}{4}x^2\right)^3} = \frac{3x^4}{8} \cdot \frac{64}{125\,x^6} = \frac{24}{125\,x^2} \xrightarrow[x \to 0^+]{} +\infty.$$

Hence g is unbounded in every neighborhood of $(0,0)$ inside $[0,1]^2$, so g is not Riemann-integrable on $[0,1]^2$.

(ii) **Failure of Fubini's equality (improper/absolute sense):** Because $I_1 = \frac{1}{8} \neq -\frac{1}{8} = I_2$, the contrapositive of Fubini's theorem confirms that g is not integrable on $[0,1]^2$ even as an absolutely convergent improper double integral (in polar coordinates near the origin, $|g(r\cos\theta, r\sin\theta)|\,r = \frac{\sin(2\theta)\,|\cos(2\theta)|}{2r}$, whose radial integral $\int_0^{r_0} \frac{dr}{r} = +\infty$ diverges).

$$I_1 = \frac{1}{8}, \qquad I_2 = -\frac{1}{8}, \qquad g \text{ is not integrable on } [0,1]^2$$

▸ **Common Pitfall:**
Never conclude that a double integral $\iint_{[0,1]^2} g(x,y)\,dx\,dy$ exists merely because one iterated integral converges to a finite number. Here both iterated integrals exist and are finite, yet the double integral itself does not exist.

✓ **Sanity Check:**
Notice the skew-symmetry $g(y,x) = \frac{yx(y^2-x^2)}{(y^2+x^2)^3} = -g(x,y)$ for all $(x,y) \in [0,1]^2$. Relabeling the dummy variables $(x,y) \leftrightarrow (y,x)$ in I_2 immediately gives $I_2 = \int_0^1 \left(\int_0^1 g(y,x)\,dy\right) dx = -\int_0^1 \left(\int_0^1 g(x,y)\,dy\right) dx = -I_1$, verifying $I_2 = -\frac{1}{8}$.

Exercise 2

1. Let $D \subset \mathbb{R}^2$ be the domain bounded by $x = y^2$ and $x = y + 2$. For a continuous function f on D, express $\iint_D f(x,y)\,dx\,dy$ in two different ways, then compute $\iint_D y\,dx\,dy$. ★★☆☆☆

Method *Horizontal vs. Decomposed Vertical Slicing of a Planar Domain and Fubini's Theorem*

> ***Why we do this (Way of Thinking):***
> Inspecting the geometry of D, every horizontal line $y = \text{const}$ between $y = -1$ and $y = 2$ enters D through the parabola $x = y^2$ on the left and exits through the line $x = y + 2$ on the right, so D is a single regular domain along the x-axis. By contrast, a vertical line $x = \text{const}$ changes its lower boundary at $x = 1$: for $0 \le x \le 1$, both the lower and upper boundaries belong to the parabola ($y = -\sqrt{x}$ and $y = +\sqrt{x}$), whereas for $1 \le x \le 4$, the lower boundary switches to the straight line $y = x - 2$. Consequently, integrating $\iint_D y\,dx\,dy$ horizontally avoids square roots and requires only a single polynomial integral.

Execution:

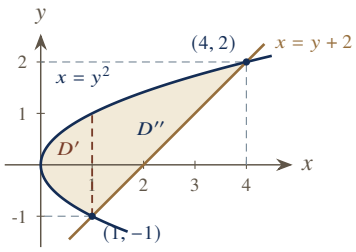

Figure 1: Domain D bounded by $x = y^2$ and $x = y + 2$, showing the vertical split line $x = 1$ ($D = D' \cup D''$).

Step 1 — Intersection points: Solving $x = y^2$ and $x = y + 2$ simultaneously yields

$$y^2 = y + 2 \iff y^2 - y - 2 = 0 \iff (y-2)(y+1) = 0 \iff y = -1 \text{ or } y = 2,$$

giving the intersection points $(1, -1)$ and $(4, 2)$, while the vertex of $x = y^2$ lies at $(0, 0)$.

Step 2 — Two iterated expressions for $\iint_D f(x,y)\,dx\,dy$:

- **First expression (regular along the x-axis, $dx\,dy$):** For all $y \in [-1, 2]$, $y^2 \le y + 2$, so

$$D = \{(x,y) \in \mathbb{R}^2 : -1 \le y \le 2,\ y^2 \le x \le y + 2\} \implies \iint_D f(x,y)\,dx\,dy = \int_{-1}^{2} \left(\int_{y^2}^{y+2} f(x,y)\,dx \right) dy.$$

- **Second expression (decomposed along the y-axis at $x = 1$, $dy\,dx$):** Splitting $D = D' \cup D''$ along $x = 1$, where $D' = \{0 \le x \le 1,\ -\sqrt{x} \le y \le \sqrt{x}\}$ and $D'' = \{1 \le x \le 4,\ x - 2 \le y \le \sqrt{x}\}$, domain additivity gives

$$\iint_D f(x,y)\,dx\,dy = \int_0^1 \left(\int_{-\sqrt{x}}^{\sqrt{x}} f(x,y)\,dy \right) dx + \int_1^4 \left(\int_{x-2}^{\sqrt{x}} f(x,y)\,dy \right) dx.$$

Step 3 — Evaluation of $\iint_D y\,dx\,dy$: Using the horizontal-slice representation:

$$\begin{aligned}
\iint_D y\,dx\,dy &= \int_{-1}^{2} y \left(\int_{y^2}^{y+2} dx \right) dy = \int_{-1}^{2} y(y + 2 - y^2)\,dy = \int_{-1}^{2} \left(-y^3 + y^2 + 2y \right) dy \\
&= \left[-\frac{y^4}{4} + \frac{y^3}{3} + y^2 \right]_{y=-1}^{y=2} = \left(-\frac{16}{4} + \frac{8}{3} + 4 \right) - \left(-\frac{1}{4} - \frac{1}{3} + 1 \right) \\
&= \left(-4 + \frac{8}{3} + 4 \right) - \left(\frac{-3 - 4 + 12}{12} \right) = \frac{8}{3} - \frac{5}{12} = \frac{32 - 5}{12} = \frac{27}{12} = \frac{9}{4}.
\end{aligned}$$

$$\iint_D f(x,y)\,dx\,dy = \int_{-1}^{2}\left(\int_{y^2}^{y+2} f(x,y)\,dx\right)dy$$
$$= \int_0^1\left(\int_{-\sqrt{x}}^{\sqrt{x}} f(x,y)\,dy\right)dx + \int_1^4\left(\int_{x-2}^{\sqrt{x}} f(x,y)\,dy\right)dx,$$
$$\iint_D y\,dx\,dy = \frac{9}{4}$$

► **Common Pitfall:**
When slicing D vertically (dy inner), starting x at $x = 1$ omits the parabolic nose $D' = \{0 \le x \le 1,\ -\sqrt{x} \le y \le \sqrt{x}\}$ to the left of the point $(1,-1)$. Always project the entire domain onto the outer axis to read the full range $x \in [0,4]$.

✓ **Sanity Check:**
Cross-checking with the second order: $\int_0^1\left(\int_{-\sqrt{x}}^{\sqrt{x}} y\,dy\right)dx = 0$ by oddness, and $\int_1^4\left[\frac{y^2}{2}\right]_{x-2}^{\sqrt{x}}dx = \frac{1}{2}\int_1^4(-x^2+5x-4)\,dx = \frac{1}{2}\left[-\frac{x^3}{3}+\frac{5x^2}{2}-4x\right]_1^4 = \frac{1}{2}\left(\frac{8}{3}+\frac{11}{6}\right) = \frac{27}{12} = \frac{9}{4}$.

2. Interchange the order of integration and compute $J = \int_0^1\left(\int_{y/2}^{y} e^{x^2}\,dx\right)dy + \int_1^2\left(\int_{y/2}^{1} e^{x^2}\,dx\right)dy$.

★★★☆☆

Method ***Domain Merging in the xy-Plane and Reversal of Integration Order***

Why we do this (Way of Thinking):
The Gaussian integrand e^{x^2} possesses no elementary antiderivative with respect to x, blocking direct evaluation of both inner dx-integrals. Furthermore, the presence of two separate integrals split at the horizontal line $y = 1$ signals that a single planar shape was sliced horizontally across an intermediate vertex at height $y = 1$. Reconstructing and merging the two pieces in the xy-plane reveals a single triangle regular along the y-axis, and integrating in y first produces an extra factor of x that unlocks e^{x^2} via $u = x^2$.

Execution: Step 1 — Reconstruction and merging of the integration domain: Write $J = \iint_{\Omega_1} e^{x^2}\,dx\,dy + \iint_{\Omega_2} e^{x^2}\,dx\,dy$, where

$$\Omega_1 = \left\{(x,y)\in\mathbb{R}^2 : 0 \le y \le 1,\ \frac{y}{2} \le x \le y\right\}, \qquad \Omega_2 = \left\{(x,y)\in\mathbb{R}^2 : 1 \le y \le 2,\ \frac{y}{2} \le x \le 1\right\}.$$

- Ω_1 is the triangle with vertices $(0,0)$, $(1,1)$, and $\left(\frac{1}{2},1\right)$, bounded on the left by $x = \frac{y}{2}$ (i.e. $y = 2x$), on the right by $x = y$ (i.e. $y = x$), and above by $y = 1$.
- Ω_2 is the triangle with vertices $\left(\frac{1}{2},1\right)$, $(1,1)$, and $(1,2)$, bounded below by $y = 1$, on the left by $x = \frac{y}{2}$ ($y = 2x$), and on the right by $x = 1$.

The two triangles join along the shared horizontal segment $y = 1$ ($\frac{1}{2} \le x \le 1$) to form the single triangle $\Omega = \Omega_1 \cup \Omega_2$ with vertices $(0,0)$, $(1,1)$, and $(1,2)$:

$$\Omega = \left\{(x,y)\in\mathbb{R}^2 : 0 \le x \le 1,\ x \le y \le 2x\right\}.$$

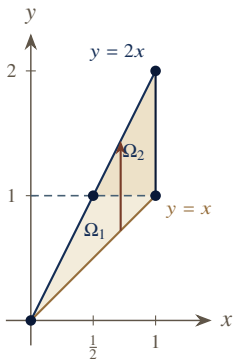

Figure 2: Merged triangular region $\Omega = \Omega_1 \cup \Omega_2 = \{0 \le x \le 1,\ x \le y \le 2x\}$ with vertical dy-slice.

Step 2 — Reversal of order and integration: Since $(x, y) \mapsto e^{x^2}$ is continuous on Ω, domain additivity and Fubini's theorem give

$$J = \iint_{\Omega} e^{x^2}\,dx\,dy = \int_0^1 \left(\int_x^{2x} e^{x^2}\,dy\right) dx = \int_0^1 e^{x^2}\Big[y\Big]_{y=x}^{y=2x} dx = \int_0^1 e^{x^2}(2x - x)\,dx = \int_0^1 x\,e^{x^2}\,dx.$$

Sub **Let $u = x^2 \implies du = 2x\,dx \implies x\,dx = \frac{1}{2}\,du$. Bounds: $x = 0 \implies u = 0$; $x = 1 \implies u = 1$.**

$$J = \frac{1}{2}\int_0^1 e^u\,du = \frac{1}{2}\Big[e^u\Big]_{u=0}^{u=1} = \frac{1}{2}(e^1 - e^0) = \frac{e-1}{2}.$$

$$\boxed{J = \int_0^1 \left(\int_x^{2x} e^{x^2}\,dy\right) dx = \frac{e-1}{2}}$$

> ▸ **Common Pitfall:**
> Inverting $x = \frac{y}{2}$ gives the upper line $y = 2x$ (not $y = \frac{x}{2}$). Reversing the inequality $\frac{y}{2} \le x \le y$ for $x, y \ge 0$ means $\frac{y}{2} \le x \iff y \le 2x$ (upper bound) and $x \le y$ (lower bound).

> ✓ **Sanity Check:**
> The triangle Ω has base $2 - 1 = 1$ along the vertical line $x = 1$ and altitude 1 from $x = 0$ to $x = 1$, so $\text{Area}(\Omega) = \frac{1}{2}(1)(1) = \frac{1}{2}$. Since $1 \le e^{x^2} \le e$ on Ω, we must have $\frac{1}{2} \le J \le \frac{e}{2} \approx 1.359$; indeed $\frac{e-1}{2} \approx 0.859 \in [0.5, 1.359]$.

3. Sketch the domain $D_1 = \left\{(x, y) \in \mathbb{R}^2 : x \ge 0,\ y \ge 0,\ 2y \le x^2 + y^2 \le 2\right\}$ and compute $\iint_{D_1} x\,e^y\,dx\,dy$.

★★★☆☆

Method ***Geometric Domain Characterization by Horizontal Slicing and Integration by Parts***

> ***Why we do this (Way of Thinking):***
> Two clues point directly to integrating with respect to x first ($dx\,dy$). Geometrically, the two circular arcs $x^2 + (y-1)^2 = 1$ (from $x^2 + y^2 = 2y$) and $x^2 + y^2 = 2$ meet in the first quadrant at $(1, 1)$, so y never exceeds 1: every horizontal slice at height $y \in [0, 1]$ enters D_1 on the circle $x = \sqrt{2y - y^2}$ and exits on the circle $x = \sqrt{2 - y^2}$ without requiring domain decomposition. Algebraically, the factor x in $x\,e^y$ integrates to $\frac{1}{2}x^2$, which squares away both radicals simultaneously and cancels $-y^2$.

Execution: Step 1 — Boundary curves, intersection, and horizontal-slice bounds:

- $x^2 + y^2 = 2$ is the circle centered at $(0, 0)$ of radius $\sqrt{2}$.
- $x^2 + y^2 = 2y \iff x^2 + (y-1)^2 = 1$ is the circle centered at $(0, 1)$ of radius 1.
- Solving $x^2 + y^2 = 2$ and $x^2 + y^2 = 2y$ simultaneously gives $2y = 2 \implies y = 1$, and $x^2 + 1 = 2 \implies x = 1$ (since $x \geq 0$), so the two circles intersect in the first quadrant at $(1, 1)$.

Notice that $2y \leq x^2 + y^2 \leq 2$ requires $2y \leq 2 \iff y \leq 1$; combined with $y \geq 0$, we obtain $0 \leq y \leq 1$. For each fixed $y \in [0, 1]$, subtracting y^2 across $2y \leq x^2 + y^2 \leq 2$ yields

$$2y - y^2 \leq x^2 \leq 2 - y^2.$$

Since $2y - y^2 = y(2 - y) \geq 0$ on $[0, 1]$ and $x \geq 0$, taking non-negative square roots characterizes D_1 as a single regular domain along the x-axis:

$$D_1 = \left\{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq 1,\ \sqrt{2y - y^2} \leq x \leq \sqrt{2 - y^2}\right\}.$$

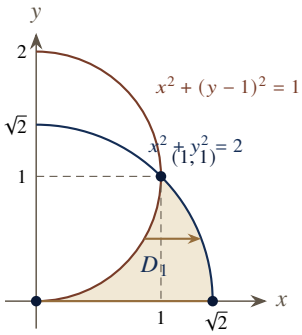

Figure 3: Domain $D_1 = \{0 \leq y \leq 1,\ \sqrt{2y - y^2} \leq x \leq \sqrt{2 - y^2}\}$ bounded below by Ox, on the left by $x^2 + (y-1)^2 = 1$, and on the right by $x^2 + y^2 = 2$.

Step 2 — Inner integration with respect to x and outer integration by parts: By Fubini's theorem on D_1:

$$\begin{aligned}\iint_{D_1} x\, e^y \, dx\, dy &= \int_0^1 e^y \left(\int_{\sqrt{2y-y^2}}^{\sqrt{2-y^2}} x \, dx \right) dy = \int_0^1 e^y \left[\frac{x^2}{2} \right]_{x=\sqrt{2y-y^2}}^{x=\sqrt{2-y^2}} dy \\ &= \frac{1}{2} \int_0^1 e^y \Big((2 - y^2) - (2y - y^2) \Big) \, dy = \frac{1}{2} \int_0^1 (2 - 2y)\, e^y \, dy = \int_0^1 (1 - y)\, e^y \, dy.\end{aligned}$$

Apply integration by parts with $u(y) = 1 - y \implies u'(y) = -1$ and $v'(y) = e^y \implies v(y) = e^y$:

$$\begin{aligned}\int_0^1 (1 - y)\, e^y \, dy &= \Big[(1 - y)\, e^y \Big]_{y=0}^{y=1} - \int_0^1 (-1)\, e^y \, dy \\ &= \left(0 \cdot e^1 - 1 \cdot e^0\right) + \Big[e^y \Big]_{y=0}^{y=1} \\ &= -1 + (e - 1) = e - 2.\end{aligned}$$

$$\boxed{\iint_{D_1} x\, e^y \, dx\, dy = e - 2}$$

> ► **Common Pitfall:**
> Do not extend the upper bound of y to $\sqrt{2}$ (the top of the outer circle $x^2 + y^2 = 2$). For every $y > 1$, the disk $x^2 + y^2 \leq 2$ lies strictly inside the disk $x^2 + y^2 < 2y$, so no point (x, y) with $y > 1$ satisfies $2y \leq x^2 + y^2 \leq 2$.

> **✓ Sanity Check:**
> Switching to polar coordinates fails here because $e^y = e^{r\sin\theta}$ creates an intractable radial integral $\int e^{r\sin\theta} r^2\, dr$, confirming that Cartesian horizontal slicing is the exact structural choice. Moreover, $e - 2 \approx 0.7183 > 0$, consistent with $x\, e^y \geq 0$ on D_1.

4. Compute, in two different ways, the integral $J = \iint_{[0,1]\times[0,1]} \frac{x}{(1+x^2)(1+xy)}\, dx\, dy$. Deduce the value of $I = \int_0^1 \frac{\ln(1+x)}{1+x^2}\, dx$.

★★★★☆

Method *Two-Order Fubini Evaluation on the Unit Square and Partial Fractions with Parameter y*

> ***Why we do this (Way of Thinking):***
> Direct single-variable integration of $I = \int_0^1 \frac{\ln(1+x)}{1+x^2}\, dx$ by parts leads to non-elementary dilogarithms. However, on the unit square $[0,1]^2$, the continuous bivariate integrand $f(x,y) = \frac{x}{(1+x^2)(1+xy)}$ can be integrated in both orders by Fubini's theorem:
>
> - Integrating with respect to y first turns $\int_0^1 \frac{x}{1+xy}\, dy$ straight into $\ln(1+x)$, giving $J = I$.
> - Integrating with respect to x first treats y as a constant parameter in a rational function of x: decomposing $\frac{x}{(1+x^2)(1+xy)}$ into partial fractions in x produces elementary ln and arctan terms plus $-I$ itself. Equating the two expressions for J isolates $2I$ and solves for both I and J.

Execution: Since $1 + x^2 \geq 1$ and $1 + xy \geq 1$ for all $(x,y) \in [0,1]^2$, the integrand $(x,y) \mapsto \frac{x}{(1+x^2)(1+xy)}$ is continuous on $[0,1]^2$. By Fubini's theorem, both iterated integrals exist and equal J.

Step 1 — First way (integrating with respect to y first, then x):

$$J = \int_0^1 \frac{1}{1+x^2} \left(\int_0^1 \frac{x}{1+xy}\, dy \right) dx.$$

For $x = 0$, the inner integral is $\int_0^1 0\, dy = 0 = \ln(1+0)$. For fixed $x \in (0,1]$, $\frac{\partial}{\partial y} \ln(1+xy) = \frac{x}{1+xy}$ (since $1 + xy > 0$), so

$$\int_0^1 \frac{x}{1+xy}\, dy = \Big[\ln(1+xy)\Big]_{y=0}^{y=1} = \ln(1+x) - \ln 1 = \ln(1+x).$$

Substituting this inner result into the outer integral gives our first relation:

$$J = \int_0^1 \frac{\ln(1+x)}{1+x^2}\, dx = I.$$

Step 2 — Second way (integrating with respect to x first via partial fractions): Fix $y \in [0,1]$ and decompose the rational function $x \mapsto \frac{x}{(1+x^2)(1+xy)}$ into partial fractions with coefficients A, B, C depending on y:

$$\frac{x}{(1+x^2)(1+xy)} = \frac{Ax+B}{1+x^2} + \frac{C}{1+xy}.$$

Clearing denominators by multiplying both sides by $(1+x^2)(1+xy)$ gives the polynomial identity in x:

$$\begin{aligned} x &= (Ax+B)(1+xy) + C(1+x^2) \\ &= Ax + Ax^2y + B + Bxy + C + Cx^2 \\ &= (Ay+C)x^2 + (A+By)x + (B+C). \end{aligned}$$

Equating coefficients of x^0, x^2, and x^1 yields the linear system:

$$\begin{cases} B + C = 0 \implies C = -B, \\ Ay + C = 0 \implies Ay - B = 0 \implies B = Ay \text{ and } C = -Ay, \\ A + By = 1 \implies A + Ay^2 = 1 \implies A(1+y^2) = 1. \end{cases}$$

Hence $A = \dfrac{1}{1+y^2}$, $B = \dfrac{y}{1+y^2}$, and $C = -\dfrac{y}{1+y^2}$, which yields the exact identity valid for all $(x, y) \in [0,1]^2$:

$$\frac{x}{(1+x^2)(1+xy)} = \frac{1}{1+y^2}\left(\frac{x}{1+x^2} + \frac{y}{1+x^2} - \frac{y}{1+xy}\right).$$

Integrating with respect to $x \in [0,1]$ at fixed $y \in [0,1]$:

$$\begin{aligned}\int_0^1 \frac{x}{(1+x^2)(1+xy)}\,dx &= \frac{1}{1+y^2}\int_0^1 \left(\frac{x}{1+x^2} + \frac{y}{1+x^2} - \frac{y}{1+xy}\right)dx \\ &= \frac{1}{1+y^2}\left[\frac{1}{2}\ln(1+x^2) + y\arctan x - \ln(1+xy)\right]_{x=0}^{x=1} \\ &= \frac{1}{1+y^2}\left(\frac{1}{2}\ln 2 + y\,\frac{\pi}{4} - \ln(1+y)\right).\end{aligned}$$

Now integrate this result with respect to y over $[0,1]$:

$$\begin{aligned}J &= \int_0^1 \frac{1}{1+y^2}\left(\frac{1}{2}\ln 2 + \frac{\pi}{4}y - \ln(1+y)\right)dy \\ &= \frac{\ln 2}{2}\int_0^1 \frac{1}{1+y^2}\,dy + \frac{\pi}{4}\int_0^1 \frac{y}{1+y^2}\,dy - \int_0^1 \frac{\ln(1+y)}{1+y^2}\,dy \\ &= \frac{\ln 2}{2}\Big[\arctan y\Big]_{y=0}^{y=1} + \frac{\pi}{4}\left[\frac{1}{2}\ln(1+y^2)\right]_{y=0}^{y=1} - I \\ &= \frac{\ln 2}{2}\cdot\frac{\pi}{4} + \frac{\pi}{4}\cdot\frac{\ln 2}{2} - I = \frac{\pi}{8}\ln 2 + \frac{\pi}{8}\ln 2 - I = \frac{\pi}{4}\ln 2 - I.\end{aligned}$$

Step 3 — Deduction of I and J: Equating the two Fubini expressions $J = I$ (Step 1) and $J = \frac{\pi}{4}\ln 2 - I$ (Step 2):

$$I = \frac{\pi}{4}\ln 2 - I \iff 2I = \frac{\pi}{4}\ln 2 \iff I = \frac{\pi}{8}\ln 2, \qquad \text{and hence} \qquad J = I = \frac{\pi}{8}\ln 2.$$

$$\boxed{J = \frac{\pi}{8}\ln 2, \qquad I = \int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\ln 2}$$

► **Common Pitfall:**
When integrating $\int_0^1 \frac{y}{1+xy}\,dx$ with respect to x, the factor y in the numerator is the exact derivative of $1+xy$ with respect to x, so the primitive is $\ln(1+xy)$ (with no extra $\frac{1}{y}$ factor). Dividing again by y would create a spurious divergence at $y = 0$.

✓ **Sanity Check:**
In the single-variable integral $I = \int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx$, substituting $x = \tan\theta$ ($\theta \in \left[0, \frac{\pi}{4}\right]$, $dx = (1+\tan^2\theta)\,d\theta$) gives $I = \int_0^{\pi/4} \ln(1+\tan\theta)\,d\theta$. Replacing θ by $\frac{\pi}{4} - \theta$ uses $1 + \tan\left(\frac{\pi}{4} - \theta\right) = \frac{2}{1+\tan\theta}$, so $I = \int_0^{\pi/4}\big(\ln 2 - \ln(1+\tan\theta)\big)\,d\theta = \frac{\pi}{4}\ln 2 - I \implies I = \frac{\pi}{8}\ln 2$, independently confirming our Fubini evaluation.

Exercise 3

Compute $\iint_D f(x,y)\,dx\,dy$ *in each of the following cases:*

(a) $f(x,y) = \cos\left(\dfrac{x-y}{x+y}\right)$**, where** D **is bounded by the lines** $x=0$**,** $y=0$**,** $x+y=1$**, and** $x+y=2$**.**

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Method ***Linear Change of Variables Aligned with the Integrand Ratio***

> ***Why we do this (Way of Thinking):***
> Both the oscillatory argument $\frac{x-y}{x+y}$ and the parallel boundary lines $x+y=1, x+y=2$ depend directly on the linear combinations $u=x-y$ and $v=x+y$. Introducing $(u,v)=(x-y,x+y)$ makes v constant on the two slanted edges ($1\le v\le 2$), turns $\cos\left(\frac{x-y}{x+y}\right)$ into $\cos\left(\frac{u}{v}\right)$, and converts the inner integral over u at fixed v into an elementary cosine primitive.

Execution: Step 1 — Linear coordinate map and Jacobian determinant:

Sub **Let** $u=x-y$ **and** $v=x+y$**. Adding and subtracting gives the inverse map** $\phi(u,v)=\left(\dfrac{v+u}{2},\dfrac{v-u}{2}\right)$**.**

The Jacobian determinant of ϕ is constant:

$$J_\phi(u,v)=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\frac{1}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{vmatrix}=\frac{1}{4}-\left(-\frac{1}{4}\right)=\frac{1}{2}\quad\Longrightarrow\quad\left|J_\phi(u,v)\right|=\frac{1}{2}.$$

Step 2 — Pullback domain $\Delta=\phi^{-1}(D)$**:** The trapezoid D is defined by $x\ge 0$, $y\ge 0$, and $1\le x+y\le 2$. Translating each boundary inequality into (u,v):

$$\begin{cases}1\le x+y\le 2 \iff 1\le v\le 2,\\ x\ge 0 \iff \dfrac{v+u}{2}\ge 0 \iff u\ge -v,\\ y\ge 0 \iff \dfrac{v-u}{2}\ge 0 \iff u\le v.\end{cases}$$

Hence $\Delta=\{(u,v)\in\mathbb{R}^2 : 1\le v\le 2,\ -v\le u\le v\}$, and $\phi:\Delta\to D$ is a C^1 bijection.

Step 3 — Change-of-variables integration: By the two-dimensional change-of-variables theorem:

$$I_a=\iint_D \cos\left(\frac{x-y}{x+y}\right)dx\,dy=\iint_\Delta \cos\left(\frac{u}{v}\right)\cdot\frac{1}{2}\,du\,dv=\frac{1}{2}\int_1^2\left(\int_{-v}^{v}\cos\left(\frac{u}{v}\right)du\right)dv.$$

Sub **For fixed** $v\in[1,2]$**, let** $t=\dfrac{u}{v}\Longrightarrow$ ***du = v dt.*** **Bounds:** $u=-v\Longrightarrow t=-1$**;** $u=v\Longrightarrow t=1$**.**

Evaluating the inner integral and then the outer integral:

$$\int_{-v}^{v}\cos\left(\frac{u}{v}\right)du=v\int_{-1}^{1}\cos t\,dt=v\Big[\sin t\Big]_{t=-1}^{t=1}=v\big(\sin 1-\sin(-1)\big)=2v\sin 1,$$

$$I_a=\frac{1}{2}\int_1^2 2v\sin 1\,dv=\sin 1\int_1^2 v\,dv=\sin 1\left[\frac{v^2}{2}\right]_{v=1}^{v=2}=\sin 1\cdot\frac{4-1}{2}=\frac{3}{2}\sin 1.$$

$$\boxed{\iint_D \cos\left(\frac{x-y}{x+y}\right)dx\,dy=\frac{3}{2}\sin 1}$$

> **► Common Pitfall:**
> Two frequent mistakes occur here: (1) using $\left|\frac{\partial(u,v)}{\partial(x,y)}\right|=2$ instead of $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=\frac{1}{2}$, and (2) forgetting the chain-rule factor v when integrating $\int\cos\left(\frac{u}{v}\right)du=v\sin\left(\frac{u}{v}\right)$.

> ✓ **Sanity Check:**
> The trapezoid D is the difference of two right triangles of legs 2 and 1, so $\text{Area}(D) = \frac{1}{2}(2^2) - \frac{1}{2}(1^2) = \frac{3}{2}$. Since $\left|\frac{x-y}{x+y}\right| \le 1$ on D, $\cos 1 \le \cos\left(\frac{x-y}{x+y}\right) \le 1$, requiring $\frac{3}{2}\cos 1 \approx 0.810 \le I_a \le 1.5$. Indeed, $\frac{3}{2}\sin 1 \approx 1.262 \in [0.810, 1.5]$.

(b) $f(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}$, **where** $D = \left\{(x,y) \in \mathbb{R}^2 : x \ge 0,\ 1 \le x^2+y^2 \le 2y\right\}$. ★★★☆☆

Method ***Polar Coordinates on an Off-Center Circular Crescent***

> ***Why we do this (Way of Thinking):***
> Both the integrand $\frac{x}{\sqrt{x^2+y^2}} = \cos\theta$ and the circular boundaries $x^2+y^2 = 1$ ($r = 1$) and $x^2+y^2 = 2y$ ($r = 2\sin\theta$) simplify immediately in polar coordinates. The key geometric step is determining the exact angular range of θ: because r must satisfy $1 \le r \le 2\sin\theta$, a valid radial interval exists only when $2\sin\theta \ge 1$, i.e. $\sin\theta \ge \frac{1}{2}$, which restricts θ inside the first quadrant ($x \ge 0, y > 0$) to $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$.

Execution: Step 1 — Polar parameterization of D:

Sub **Pass to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ ($r > 0$), with Jacobian $|J(r,\theta)| = r$ and $dx\,dy = r\,dr\,d\theta$.**

Translating the inequalities defining D:

- $1 \le x^2+y^2 \le 2y \iff 1 \le r^2 \le 2r\sin\theta$. Since $r \ge 1 > 0$, dividing $r^2 \le 2r\sin\theta$ by r yields $1 \le r \le 2\sin\theta$.
- For the radial interval $[1, 2\sin\theta]$ to be non-empty, we must have $2\sin\theta \ge 1 \iff \sin\theta \ge \frac{1}{2}$ (which in particular forces $y = r\sin\theta > 0$). Combined with $x = r\cos\theta \ge 0 \iff \cos\theta \ge 0$, θ lies in $\left[0, \frac{\pi}{2}\right]$ with $\sin\theta \ge \frac{1}{2}$, hence
$$\frac{\pi}{6} \le \theta \le \frac{\pi}{2}.$$

Thus $\Delta = \left\{(r,\theta) : \frac{\pi}{6} \le \theta \le \frac{\pi}{2},\ 1 \le r \le 2\sin\theta\right\}$.

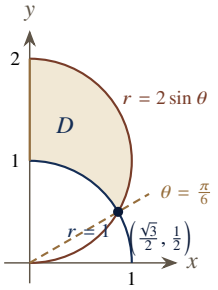


Figure 4: Polar region $D = \left\{\frac{\pi}{6} \le \theta \le \frac{\pi}{2},\ 1 \le r \le 2\sin\theta\right\}$ between $x^2+y^2 = 1$ and $x^2+(y-1)^2 = 1$.

Step 2 — Evaluation of the polar double integral: Transforming the integrand together with the Jacobian r:
$$f(r\cos\theta, r\sin\theta)\, r = \frac{r\cos\theta}{r} \cdot r = r\cos\theta.$$

Integrating with respect to r first and then θ:
$$I_b = \int_{\pi/6}^{\pi/2} \cos\theta \left(\int_1^{2\sin\theta} r\,dr\right) d\theta = \int_{\pi/6}^{\pi/2} \cos\theta \left[\frac{r^2}{2}\right]_{r=1}^{r=2\sin\theta} d\theta$$
$$= \frac{1}{2}\int_{\pi/6}^{\pi/2} \left(4\sin^2\theta - 1\right)\cos\theta\, d\theta.$$

Sub **Let $u = \sin\theta \implies du = \cos\theta\, d\theta$. Bounds: $\theta = \frac{\pi}{6} \implies u = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$; $\theta = \frac{\pi}{2} \implies u = \sin\left(\frac{\pi}{2}\right) = 1$.**

$$I_b = \frac{1}{2}\int_{1/2}^{1}(4u^2-1)\,du = \frac{1}{2}\left[\frac{4u^3}{3} - u\right]_{u=1/2}^{u=1}$$
$$= \frac{1}{2}\left[\left(\frac{4}{3}-1\right) - \left(\frac{4}{3}\cdot\frac{1}{8} - \frac{1}{2}\right)\right] = \frac{1}{2}\left[\frac{1}{3} - \left(\frac{1}{6} - \frac{3}{6}\right)\right] = \frac{1}{2}\left(\frac{1}{3} + \frac{1}{3}\right) = \frac{1}{3}.$$

$$\boxed{\iint_D \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy = \frac{1}{3}}$$

> ► **Common Pitfall:**
> Assuming $\theta \in \left[0, \frac{\pi}{2}\right]$ merely because $x \geq 0$ and $y \geq 0$ is incorrect: for $\theta \in \left[0, \frac{\pi}{6}\right)$, $2\sin\theta < 1$, which reverses the bounds (r would run backward from 1 to $2\sin\theta < 1$) and adds a negative fictitious region outside D.

> ✓ **Sanity Check:**
> Check the Cartesian intersection of $x^2+y^2 = 1$ and $x^2+y^2 = 2y$: $2y = 1 \implies y = \frac{1}{2}, x = \sqrt{1-\frac{1}{4}} = \frac{\sqrt{3}}{2}$, whose polar angle satisfies $\tan\theta = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}$, confirming $\theta = \frac{\pi}{6}$.

(c) $f(x,y) = (y^2-x^2)^{xy}(x^2+y^2)$**, where** $D = \left\{(x,y) \in \mathbb{R}^2 : 0 < x < y,\ 1 < xy < 2,\ y^2 - x^2 < 1\right\}$**.**

★★★★☆

Method ***Curvilinear Change of Variables via the Inverse Jacobian Theorem and Order Choice***

> ***Why we do this (Way of Thinking):***
> Look simultaneously at the boundary inequalities ($1 < xy < 2$ and $0 < y^2 - x^2 < 1$) and the base-exponent factor $(y^2-x^2)^{xy}$: setting $u = y^2-x^2$ and $v = xy$ maps D onto the rectangle $\Delta = (0,1)\times(1,2)$. Moreover, the forward Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$ equals $-2(x^2+y^2)$, so its reciprocal $|J_\phi| = \frac{1}{2(x^2+y^2)}$ cancels the factor (x^2+y^2) in the integrand. Once reduced to $\frac{1}{2}\iint_{(0,1)\times(1,2)} u^v\,du\,dv$, we integrate with respect to u first (power rule) rather than v first (which would introduce $\frac{1}{\ln u}$).

Execution: Step 1 — Bijectivity of the coordinate transformation on D:

Sub **Define $\psi(x,y) = (u,v)$ by $u = y^2 - x^2$ and $v = xy$ on D.**

On D, $0 < x < y$ implies $y^2 - x^2 > 0$, so $0 < u < 1$ and $1 < v < 2$, mapping D into the open rectangle $\Delta = (0,1)\times(1,2)$. Conversely, for every $(u,v) \in \Delta$, using the algebraic identity

$$(x^2+y^2)^2 = (y^2-x^2)^2 + 4x^2y^2 = u^2 + 4v^2,$$

positivity of x^2+y^2 gives $x^2+y^2 = \sqrt{u^2+4v^2}$. Adding and subtracting $u = y^2-x^2$ uniquely determines

$$x = \sqrt{\frac{\sqrt{u^2+4v^2}-u}{2}} > 0, \qquad y = \sqrt{\frac{\sqrt{u^2+4v^2}+u}{2}} > x > 0,$$

establishing that $\phi = \psi^{-1} : \Delta \to D$ is a C^1 bijection.

Step 2 — Jacobian via the inverse Jacobian determinant: Compute the Jacobian of $\psi(x,y) = (y^2 - x^2, xy)$:

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -2x & 2y \\ y & x \end{vmatrix} = -2x^2 - 2y^2 = -2(x^2+y^2) < 0 \quad \text{on } D.$$

By the inverse function theorem, the Jacobian of $\phi = \psi^{-1}$ satisfies

$$\left|J_\phi(u,v)\right| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|} = \frac{1}{2(x^2+y^2)}.$$

Step 3 — Change of variables and iterated integration (du first): Multiplying the integrand by $|J_\phi(u,v)|$ cancels (x^2+y^2):

$$f\big(\phi(u,v)\big)\left|J_\phi(u,v)\right| = u^v\,(x^2+y^2)\cdot\frac{1}{2(x^2+y^2)} = \frac{1}{2}u^v.$$

Applying Fubini's theorem on $\Delta = (0,1)\times(1,2)$ by integrating with respect to $u \in (0,1)$ first (since $v > 1 > -1$):

$$\begin{aligned} I_c &= \iint_{(0,1)\times(1,2)} \frac{1}{2}u^v\,du\,dv = \frac{1}{2}\int_1^2 \left(\int_0^1 u^v\,du\right)dv = \frac{1}{2}\int_1^2 \left[\frac{u^{v+1}}{v+1}\right]_{u=0}^{u=1} dv \\ &= \frac{1}{2}\int_1^2 \frac{1-0}{v+1}\,dv = \frac{1}{2}\Big[\ln(v+1)\Big]_{v=1}^{v=2} = \frac{1}{2}(\ln 3 - \ln 2) = \frac{1}{2}\ln\left(\frac{3}{2}\right). \end{aligned}$$

$$\boxed{\iint_D (y^2-x^2)^{xy}\,(x^2+y^2)\,dx\,dy = \frac{1}{2}\ln\left(\frac{3}{2}\right)}$$

> ▸ **Common Pitfall:**
> Never try to invert $u = y^2 - x^2, v = xy$ first and differentiate those nested square roots to find $\frac{\partial(x,y)}{\partial(u,v)}$ directly: computing $\frac{\partial(u,v)}{\partial(x,y)} = -2(x^2+y^2)$ and taking its reciprocal saves pages of algebra and displays the exact (x^2+y^2) cancellation immediately.

> ✓ **Sanity Check:**
> On $\Delta = (0,1)\times(1,2)$, $0 < u^2 < u^v < u^1$ since $u \in (0,1)$ and $v \in (1,2)$. Integrating $\frac{1}{2}u^2$ and $\frac{1}{2}u$ over Δ bounds I_c between $\frac{1}{2}\cdot\frac{1}{3}\cdot 1 = \frac{1}{6} \approx 0.1667$ and $\frac{1}{2}\cdot\frac{1}{2}\cdot 1 = \frac{1}{4} = 0.25$; indeed $\frac{1}{2}\ln(1.5) \approx 0.2027 \in (0.1667, 0.25)$.

(d) $f(x,y) = (y^2-x^2)\cos(xy)$**, where** $D = \{(x,y)\in\mathbb{R}^2 : 0 \le x \le y,\ x+y \le 4,\ xy \ge 1\}$**.** ★★★★☆

Method ***Symmetric Sum-Product Change of Variables ($u = x+y$, $v = xy$) via Vieta's Discriminant Condition***

> ***Why we do this (Way of Thinking):***
> Factoring $y^2 - x^2 = (y-x)(x+y)$ reveals the three building blocks of the problem: $x+y$ (bounded above by 4), xy (bounded below by 1 and appearing inside $\cos(xy)$), and $y-x$. Setting $u = x+y$ and $v = xy$ has forward Jacobian $\frac{\partial(u,v)}{\partial(x,y)} = x - y = -(y-x)$, so $|J_\phi| = \frac{1}{y-x}$ cancels $(y-x)$ from $(y-x)(x+y)\cos(xy)$, leaving $u\cos v$. The delicate step is finding the exact image of the diagonal boundary $x = y$: by Vieta's theorem, x and y are real roots of $t^2 - ut + v = 0$ with $x \le y$ if and only if the discriminant satisfies $u^2 - 4v \ge 0$, so $x = y$ maps to the parabola $v = \frac{u^2}{4}$ (rather than a rectangle).

Execution: Step 1 — Bijectivity on the interior D° and exact pullback domain Δ: Since boundary curves have zero planar area, we work on the interior

$$D^\circ = \{(x,y) \in \mathbb{R}^2 : 0 < x < y,\ x+y < 4,\ xy > 1\}$$

(noting that $xy \geq 1$ and $0 \leq x \leq y$ already prevent $x = 0$).

Sub **Define $\psi(x,y) = (u,v)$ by $u = x+y$ and $v = xy$.**

For any $(x,y) \in D^\circ$, x and y are the two distinct positive roots ($0 < x < y$) of the quadratic equation $t^2 - ut + v = 0$, namely

$$x = \frac{u - \sqrt{u^2 - 4v}}{2}, \qquad y = \frac{u + \sqrt{u^2 - 4v}}{2}.$$

A pair $(u,v) \in \mathbb{R}^2$ corresponds to a unique point $(x,y) \in D^\circ$ if and only if:

(i) $v = xy > 1$ and $u = x + y < 4$ (in particular $u > 0$ and $v > 0$, which automatically ensures $x = \frac{u-\sqrt{u^2-4v}}{2} > 0$ whenever $u^2 - 4v > 0$), and

(ii) the discriminant is strictly positive: $u^2 - 4v > 0 \iff v < \dfrac{u^2}{4}$ (corresponding to $x < y$).

Requiring $1 < v < \frac{u^2}{4}$ forces $\frac{u^2}{4} > 1 \iff u > 2$ (since $u > 0$). Thus the closure of the pullback domain is the curvilinear triangle

$$\Delta = \left\{(u,v) \in \mathbb{R}^2 : 2 \leq u \leq 4,\ 1 \leq v \leq \frac{u^2}{4}\right\}.$$

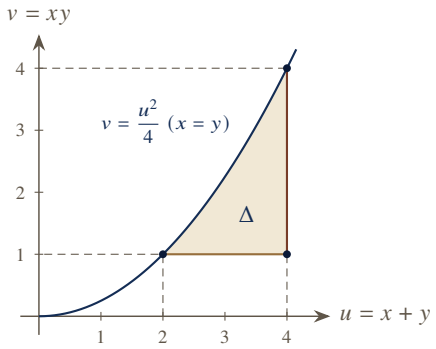


Figure 5: Pullback domain $\Delta = \left\{2 \leq u \leq 4,\ 1 \leq v \leq \frac{u^2}{4}\right\}$ in the (u,v)-plane.

Step 2 — Jacobian and integrand simplification: For $\psi(x,y) = (x+y, xy)$ on D° ($y > x$):

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ y & x \end{vmatrix} = x - y < 0 \quad \Longrightarrow \quad \left|J_\phi(u,v)\right| = \frac{1}{|x-y|} = \frac{1}{y-x}.$$

Transforming the integrand together with $|J_\phi(u,v)|$:

$$f(x,y)\left|J_\phi(u,v)\right| = (y^2 - x^2)\cos(xy) \cdot \frac{1}{y-x} = (y-x)(x+y)\cos(xy) \cdot \frac{1}{y-x} = (x+y)\cos(xy) = u\cos v.$$

Step 3 — Iterated integration on Δ: Integrating with respect to v first and then u:

$$I_d = \iint_\Delta u\cos v\, du\, dv = \int_2^4 u\left(\int_1^{u^2/4} \cos v\, dv\right) du = \int_2^4 u\Big[\sin v\Big]_{v=1}^{v=u^2/4} du$$
$$= \int_2^4 u\sin\left(\frac{u^2}{4}\right) du - \sin 1 \int_2^4 u\, du.$$

Sub **In the first integral, let** $w = \dfrac{u^2}{4} \implies \boldsymbol{dw = \frac{u}{2}\,du} \implies u\,du = 2\,dw$**. Bounds:** $u = 2 \implies w = 1$**;** $u = 4 \implies w = 4$**.**

$$\begin{aligned} I_d &= 2\int_1^4 \sin w\,dw - \sin 1\left[\frac{u^2}{2}\right]_{u=2}^{u=4} \\ &= 2\Big[-\cos w\Big]_{w=1}^{w=4} - \sin 1 \cdot \frac{16-4}{2} \\ &= 2(\cos 1 - \cos 4) - 6\sin 1. \end{aligned}$$

$$\boxed{\iint_D (y^2 - x^2)\cos(xy)\,dx\,dy = 2\cos 1 - 2\cos 4 - 6\sin 1}$$

> ▸ **Common Pitfall:**
> Writing Δ as a rectangle $[2,4]\times[1,4]$ ignores the constraint $0 \le x \le y$: for example, $(u,v) = (2,4)$ would require $x + y = 2$ and $xy = 4$, which has no real solutions ($t^2 - 2t + 4 = (t-1)^2 + 3 > 0$). Always check the discriminant $u^2 - 4v \ge 0$ when mapping symmetric polynomials $(x + y, xy)$.

> ✓ **Sanity Check:**
> Reversing the integration order on $\Delta = \{1 \le v \le 4,\ 2\sqrt{v} \le u \le 4\}$ gives $\int_1^4 \cos v \left[\frac{u^2}{2}\right]_{2\sqrt{v}}^4 dv = \int_1^4 (8 - 2v)\cos v\,dv$. Integrating by parts yields $\Big[(8-2v)\sin v\Big]_1^4 + 2\int_1^4 \sin v\,dv = -6\sin 1 + 2(\cos 1 - \cos 4)$, matching our result identically.

Exercise 4

1. Compute the area of the planar domain D bounded by the curves $y = ax^2$, $y = bx^2$, $xy = c$, and $xy = d$ $(0 < a < b, 0 < c < d, x > 0, y > 0)$. ★★★☆☆

Method ***Planar Area via Curvilinear Coordinate Straightening***

> ***Why we do this (Way of Thinking):***
> In Cartesian coordinates, the four curved boundaries $y = ax^2$, $y = bx^2$, $y = \frac{c}{x}$, and $y = \frac{d}{x}$ intersect at four distinct x-coordinates, which would split $\text{Area}(D) = \iint_D dx\,dy$ into three separate integrals. Rewriting the two curve families as constant-level curves $\frac{y}{x^2} = \text{const}$ and $xy = \text{const}$ straightens D into the single rectangle $\Delta = [a,b]\times[c,d]$ in one step.

Execution: Step 1 — Curvilinear change of variables and inverse map:

Sub **On** $x > 0, y > 0$**, set** $u = \dfrac{y}{x^2}$ **and** $v = xy$**, so** $(x,y) \in D \iff (u,v) \in \Delta = [a,b]\times[c,d]$**.**

To solve explicitly for (x,y) in terms of $(u,v) \in [a,b]\times[c,d]$ (where $u, v > 0$), observe that

$$uv^2 = \frac{y}{x^2}\cdot x^2y^2 = y^3 \implies y = u^{1/3}v^{2/3}, \qquad x = \frac{v}{y} = \frac{v}{u^{1/3}v^{2/3}} = u^{-1/3}v^{1/3}.$$

Thus $\phi(u,v) = \left(u^{-1/3}v^{1/3},\ u^{1/3}v^{2/3}\right)$ is a C^1 bijection from $\Delta = [a,b]\times[c,d]$ onto D.

Step 2 — Jacobian determinant: Differentiating $x(u,v) = u^{-1/3}v^{1/3}$ and $y(u,v) = u^{1/3}v^{2/3}$:

$$\begin{aligned} J_\phi(u,v) = \frac{\partial(x,y)}{\partial(u,v)} &= \begin{vmatrix} -\frac{1}{3}u^{-4/3}v^{1/3} & \frac{1}{3}u^{-1/3}v^{-2/3} \\ \frac{1}{3}u^{-2/3}v^{2/3} & \frac{2}{3}u^{1/3}v^{-1/3} \end{vmatrix} \\ &= \left(-\frac{1}{3}u^{-4/3}v^{1/3}\right)\left(\frac{2}{3}u^{1/3}v^{-1/3}\right) - \left(\frac{1}{3}u^{-1/3}v^{-2/3}\right)\left(\frac{1}{3}u^{-2/3}v^{2/3}\right) \\ &= -\frac{2}{9}u^{-1} - \frac{1}{9}u^{-1} = -\frac{3}{9u} = -\frac{1}{3u}, \end{aligned}$$

so $\left|J_\phi(u,v)\right| = \dfrac{1}{3u}$ on Δ (since $u \geq a > 0$).

Step 3 — Area evaluation: Applying the change-of-variables formula to $\text{Area}(D) = \iint_D dx\,dy$:

$$\begin{aligned} \text{Area}(D) &= \iint_{[a,b]\times[c,d]} \frac{1}{3u}\,du\,dv = \left(\int_a^b \frac{1}{3u}\,du\right)\left(\int_c^d dv\right) \\ &= \frac{1}{3}\Big[\ln u\Big]_{u=a}^{u=b} \cdot \Big[v\Big]_{v=c}^{v=d} = \frac{1}{3}(\ln b - \ln a)(d-c) = \frac{d-c}{3}\ln\left(\frac{b}{a}\right). \end{aligned}$$

$$\boxed{\text{Area}(D) = \frac{d-c}{3}\ln\left(\frac{b}{a}\right)}$$

> ▸ **Common Pitfall:**
> Writing $u = y - ax^2$ fails because a and b are distinct parameters: you must isolate the parameter on the right-hand side ($y = kx^2 \iff \frac{y}{x^2} = k$) so that both boundaries become constant coordinate lines $u = a$ and $u = b$.

> ✓ **Sanity Check:**
> Verify J_ϕ via the forward Jacobian: $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -2y/x^3 & 1/x^2 \\ y & x \end{vmatrix} = -\frac{2y}{x^2} - \frac{y}{x^2} = -\frac{3y}{x^2} = -3u$, whose reciprocal magnitude is indeed $\frac{1}{3u}$. Also, $\text{Area}(D) > 0$ since $d > c$ and $b > a > 0 \implies \ln(b/a) > 0$, and $\text{Area}(D) = 0$ if $a = b$ or $c = d$.

2. Compute the mass M, the average density Q_M, and the center of gravity (x_G, y_G) of the plate $D = \{(x,y) \in \mathbb{R}^2 : y \geq 0,\ 1 \leq x^2 + y^2 \leq 4\}$ with density $\rho(x,y) = y\sqrt{x^2+y^2}$. ★★☆☆☆

Method ***Planar Mass, Average Density, Parity Symmetry, and Center of Gravity in Polar Coordinates***

> ***Why we do this (Way of Thinking):***
> The upper half-annulus D becomes the rectangle $[1,2] \times [0,\pi]$ in polar coordinates, and the density $\rho(x,y) = y\sqrt{x^2+y^2}$ becomes the separable monomial $r^2\sin\theta$. Before computing x_G, notice that D is symmetric across the y-axis while $\rho(-x,y) = \rho(x,y)$ is even in x: therefore $x\,\rho(x,y)$ is odd in x, immediately giving $x_G = 0$.

Execution:

Sub **Pass to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$. On D, $1 \leq r^2 \leq 4 \iff 1 \leq r \leq 2$ and $y \geq 0 \iff 0 \leq \theta \leq \pi$, so $\Delta = [1,2]\times[0,\pi]$ and $\rho(r\cos\theta, r\sin\theta) = (r\sin\theta)\,r = r^2\sin\theta$.**

Step 1 — Area and total mass M: The area of the upper half-annulus between radii 1 and 2 is

$$\text{Area}(D) = \iint_D dx\,dy = \int_0^\pi d\theta \int_1^2 r\,dr = \pi\left[\frac{r^2}{2}\right]_1^2 = \frac{3\pi}{2}.$$

The total mass M is

$$M = \iint_D \rho(x,y)\,dx\,dy = \iint_{[1,2]\times[0,\pi]} (r^2\sin\theta)\,r\,dr\,d\theta = \left(\int_0^{\pi} \sin\theta\,d\theta\right)\left(\int_1^2 r^3\,dr\right)$$
$$= \Big[-\cos\theta\Big]_{\theta=0}^{\theta=\pi} \cdot \left[\frac{r^4}{4}\right]_{r=1}^{r=2} = \big(-(-1)-(-1)\big)\cdot\frac{16-1}{4} = 2\cdot\frac{15}{4} = \frac{15}{2}.$$

Step 2 — Average density Q_M:

$$Q_M = \frac{M}{\text{Area}(D)} = \frac{15/2}{3\pi/2} = \frac{15}{3\pi} = \frac{5}{\pi}.$$

Step 3 — Center of gravity (x_G, y_G):

- **Abscissa x_G:** For every $(x,y)\in D$, $(-x,y)\in D$ and $(-x)\rho(-x,y) = -x\rho(x,y)$ (equivalently, in polar coordinates $\int_0^{\pi}\cos\theta\sin\theta\,d\theta = \left[\frac{\sin^2\theta}{2}\right]_0^{\pi} = 0$). Hence $\iint_D x\,\rho(x,y)\,dx\,dy = 0$, so $x_G = 0$.
- **Ordinate y_G:** Using the power-reduction identity $\sin^2\theta = \dfrac{1-\cos(2\theta)}{2}$:

$$\iint_D y\,\rho(x,y)\,dx\,dy = \int_0^{\pi}\int_1^2 (r\sin\theta)(r^2\sin\theta)\,r\,dr\,d\theta = \left(\int_0^{\pi}\sin^2\theta\,d\theta\right)\left(\int_1^2 r^4\,dr\right)$$
$$= \left(\frac{1}{2}\int_0^{\pi}\big(1-\cos(2\theta)\big)\,d\theta\right)\left[\frac{r^5}{5}\right]_{r=1}^{r=2}$$
$$= \frac{1}{2}\left[\theta - \frac{\sin(2\theta)}{2}\right]_{\theta=0}^{\theta=\pi}\cdot\frac{32-1}{5} = \frac{\pi}{2}\cdot\frac{31}{5} = \frac{31\pi}{10}.$$

Dividing by $M = \frac{15}{2}$ gives

$$y_G = \frac{1}{M}\iint_D y\,\rho(x,y)\,dx\,dy = \frac{2}{15}\cdot\frac{31\pi}{10} = \frac{31\pi}{75}.$$

$$\boxed{M = \frac{15}{2}, \qquad Q_M = \frac{5}{\pi}, \qquad (x_G, y_G) = \left(0, \frac{31\pi}{75}\right)}$$

► **Common Pitfall:**
Do not divide $\iint_D y\,\rho\,dx\,dy$ by $\text{Area}(D)$ when computing y_G for a non-homogeneous lamina: the denominator in $y_G = \frac{1}{M}\iint_D y\,\rho\,dx\,dy$ is the total mass $M = \frac{15}{2}$, whereas $\text{Area}(D) = \frac{3\pi}{2}$ is the denominator of the average density Q_M.

✓ **Sanity Check:**
On D, $0 \le y \le 2$, so y_G must lie strictly inside $(0,2)$. Numerically, $y_G = \frac{31\pi}{75} \approx 1.2985 \in (0,2)$ (and lies slightly above the homogeneous half-annulus centroid $\frac{4(2^3-1^3)}{3\pi(2^2-1^2)} = \frac{28}{9\pi} \approx 0.990$ because $\rho = yr$ concentrates mass at larger y and r).

3. Let P be a homogeneous plate of mass M obtained by removing the disk $x^2+y^2 < R^2$ from the square

$[-2R, 2R]^2$ $(R > 0)$**. Express the moments of inertia I_O, I_x, and I_y of P in terms of M and R.** ★★★☆☆

Method ***Domain Subtraction of Moments of Inertia and the Perpendicular Axis Theorem***

> ***Why we do this (Way of Thinking):***
> Integrating directly over the perforated region $P = [-2R, 2R]^2 \setminus \{x^2 + y^2 < R^2\}$ in either Cartesian or polar coordinates would require splitting P into four curved pieces with inverse trigonometric integrals. Instead, by domain additivity of double integrals, the full square $S = [-2R, 2R]^2$ is the disjoint union $S = P \sqcup B$ with the central disk $B = \{x^2 + y^2 < R^2\}$, so $I(P) = I(S) - I(B)$. Note that both S and B must be integrated with the *same* constant surface density $\rho_0 = \frac{M}{\text{Area}(P)}$ before expressing the final answers in terms of the perforated plate's mass M.

Execution: Step 1 — Surface density ρ_0 in terms of M and R: Let $S = [-2R, 2R]^2$ (square of side length $4R$) and $B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < R^2\}$ (disk of radius R). Since $B \subset S$, the area of $P = S \setminus B$ is

$$\text{Area}(P) = \text{Area}(S) - \text{Area}(B) = (4R)^2 - \pi R^2 = (16 - \pi)R^2.$$

Denoting by $\rho_0 > 0$ the constant surface density of P, the total mass $M = \rho_0 \text{ Area}(P)$ gives

$$\rho_0 = \frac{M}{(16 - \pi)R^2}.$$

Step 2 — Moments of inertia of the full square S (with density ρ_0): On $S = [-2R, 2R] \times [-2R, 2R]$:

$$\begin{aligned} I_x(S) &= \iint_S y^2 \rho_0 \, dx \, dy = \rho_0 \left(\int_{-2R}^{2R} dx \right) \left(\int_{-2R}^{2R} y^2 \, dy \right) \\ &= \rho_0 \cdot (4R) \cdot \left[\frac{y^3}{3} \right]_{-2R}^{2R} = 4\rho_0 R \cdot \frac{8R^3 - (-8R^3)}{3} = \frac{64}{3} \rho_0 R^4. \end{aligned}$$

By symmetry of S under $(x, y) \leftrightarrow (y, x)$, $I_y(S) = I_x(S) = \frac{64}{3} \rho_0 R^4$, and by the Perpendicular Axis Theorem,

$$I_O(S) = I_x(S) + I_y(S) = \frac{128}{3} \rho_0 R^4.$$

Step 3 — Moments of inertia of the removed disk B (with the same density ρ_0):

Sub **On B, pass to polar coordinates $x = r \cos\theta$, $y = r \sin\theta$ $(0 \le r < R, 0 \le \theta \le 2\pi)$, $dx\,dy = r\,dr\,d\theta$.**

Computing the polar moment $I_O(B)$ directly:

$$I_O(B) = \iint_B (x^2 + y^2) \rho_0 \, dx \, dy = \rho_0 \left(\int_0^{2\pi} d\theta \right) \left(\int_0^R r^2 \cdot r \, dr \right) = 2\pi\rho_0 \left[\frac{r^4}{4} \right]_0^R = \frac{\pi}{2} \rho_0 R^4.$$

By rotational symmetry of B, $I_x(B) = I_y(B)$, and since $I_O(B) = I_x(B) + I_y(B) = 2I_x(B)$, we obtain

$$I_x(B) = I_y(B) = \frac{\pi}{4} \rho_0 R^4.$$

Step 4 — Subtraction by domain additivity and substitution of ρ_0: Since $S = P \sqcup B$:

$$I_O(P) = I_O(S) - I_O(B) = \left(\frac{128}{3} - \frac{\pi}{2} \right) \rho_0 R^4 = \frac{256 - 3\pi}{6} \cdot \frac{M}{(16 - \pi)R^2} \cdot R^4 = \frac{256 - 3\pi}{6(16 - \pi)} MR^2,$$

$$I_x(P) = I_x(S) - I_x(B) = \left(\frac{64}{3} - \frac{\pi}{4} \right) \rho_0 R^4 = \frac{256 - 3\pi}{12} \cdot \frac{M}{(16 - \pi)R^2} \cdot R^4 = \frac{256 - 3\pi}{12(16 - \pi)} MR^2,$$

$$I_y(P) = I_y(S) - I_y(B) = I_x(P) = \frac{256 - 3\pi}{12(16 - \pi)} MR^2.$$

$$I_O = \frac{256-3\pi}{6(16-\pi)}MR^2, \qquad I_x = I_y = \frac{256-3\pi}{12(16-\pi)}MR^2$$

► **Common Pitfall:**
A frequent exam error on perforated plates is writing $I_x(S) = \frac{1}{12}M(4R)^2$ and $I_x(B) = \frac{1}{4}MR^2$ with the *same* symbol M and subtracting them. Here M is the mass of the perforated plate P, whereas the unperforated square would have mass $M_S = \frac{16}{16-\pi}M$ and the removed hole would have mass $M_B = \frac{\pi}{16-\pi}M$. Always work through the common surface density ρ_0 first, then eliminate ρ_0 at the very end.

✓ **Sanity Check:**
Check two limiting checks: (1) $I_x + I_y = 2 \cdot \frac{256-3\pi}{12(16-\pi)}MR^2 = \frac{256-3\pi}{6(16-\pi)}MR^2 = I_O$, satisfying the Perpendicular Axis Theorem; (2) dropping the π terms (vanishing hole limit) gives $I_x \to \frac{256}{12\times16}MR^2 = \frac{4}{3}MR^2 = \frac{1}{12}M(4R)^2$, recovering the canonical formula $\frac{1}{12}Mh^2$ for a solid square of side $h = 4R$.

4. Compute the moment of inertia I_Δ of a homogeneous triangular plate $D = \{(x,y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 0,\ x+y \leq a\}$ ($a > 0$, density $\rho_0 > 0$) with respect to the line $\Delta : x + y = a$. ★★★☆☆

Method *Planar Moment of Inertia via Perpendicular Point-to-Line Distance*

Why we do this (Way of Thinking):
By definition, the moment of inertia of a planar domain D about any straight line Δ is $I_\Delta = \iint_D \rho(x,y)\, d^2(P,\Delta)\, dx\, dy$, where $d(P,\Delta)$ is the perpendicular Euclidean distance from $P = (x,y)$ to Δ. For a general line $\alpha x + \beta y + \gamma = 0$, recall from analytic geometry that $d(P,\Delta) = \frac{|\alpha x+\beta y+\gamma|}{\sqrt{\alpha^2+\beta^2}}$. Once we write $d^2(P,\Delta) = \frac{(a-x-y)^2}{2}$, the inner integral over $y \in [0, a-x]$ collapses cleanly to $\frac{(a-x)^3}{3}$ via a single shift substitution.

Execution: Step 1 — Squared Euclidean distance from $P(x,y) \in D$ to Δ: Rewrite the hypotenuse line Δ in implicit form $x + y - a = 0$ ($\alpha = 1, \beta = 1, \gamma = -a$). For any point $P = (x,y) \in D$, since $x + y \leq a$, the perpendicular distance to Δ is

$$d(P,\Delta) = \frac{|x+y-a|}{\sqrt{1^2+1^2}} = \frac{a-x-y}{\sqrt{2}} \quad \Longrightarrow \quad d^2(P,\Delta) = \frac{(a-x-y)^2}{2}.$$

Step 2 — Double integral over the right triangle $D = \{0 \leq x \leq a,\ 0 \leq y \leq a-x\}$:

$$I_\Delta = \iint_D \rho_0\, d^2(P,\Delta)\, dx\, dy = \frac{\rho_0}{2}\int_0^a \left(\int_0^{a-x} (a-x-y)^2\, dy\right) dx.$$

Sub **For fixed $x \in [0,a]$, let $u = a - x - y \implies$ $du = -dy$. Bounds: $y = 0 \implies u = a - x$; $y = a - x \implies u = 0$.**

Evaluating the inner y-integral:

$$\int_0^{a-x} (a-x-y)^2\, dy = -\int_{a-x}^0 u^2\, du = \int_0^{a-x} u^2\, du = \left[\frac{u^3}{3}\right]_{u=0}^{u=a-x} = \frac{(a-x)^3}{3}.$$

Substituting into the outer x-integral:

Sub **Let $w = a - x \implies$ $dw = -dx$. Bounds: $x = 0 \implies w = a$; $x = a \implies w = 0$.**

$$I_\Delta = \frac{\rho_0}{2}\int_0^a \frac{(a-x)^3}{3}\, dx = \frac{\rho_0}{6}\int_0^a w^3\, dw = \frac{\rho_0}{6}\left[\frac{w^4}{4}\right]_{w=0}^{w=a} = \frac{\rho_0 a^4}{24}.$$

$$I_\Delta = \frac{\rho_0 a^4}{24}$$

► **Common Pitfall:**
Omitting the normalization denominator $\sqrt{1^2+1^2} = \sqrt{2}$ in $d(P,\Delta) = \frac{|x+y-a|}{\sqrt{2}}$ (i.e. writing $d^2 = (a-x-y)^2$) overestimates I_Δ by a factor of 2.

✓ **Sanity Check:**
Check dimensions and classical triangle inertia: $[\rho_0] = \mathrm{M L^{-2}}$ and $[a^4] = \mathrm{L}^4$, so $[\rho_0 a^4] = \mathrm{M L^2}$. Moreover, the triangle has mass $M = \frac{1}{2}\rho_0 a^2$ and altitude $h = \frac{a}{\sqrt{2}}$ perpendicular to its hypotenuse, so the standard mechanics formula $\frac{1}{6}Mh^2 = \frac{1}{6}\left(\frac{1}{2}\rho_0 a^2\right)\left(\frac{a^2}{2}\right) = \frac{\rho_0 a^4}{24}$ agrees identically.

Exercise 5

Compute $\iiint_D f(x,y,z)\,dx\,dy\,dz$ *in each of the following cases:*

(a) $f(x,y,z) = \dfrac{1}{(1+x+y+z)^3}$, $D = \{(x,y,z) \in \mathbb{R}^3 : x \geq 0,\ y \geq 0,\ z \geq 0,\ x+y+z \leq 1\}$. ★★☆☆☆

Method *Fubini's Theorem on the Standard Unit Tetrahedron via Successive Shift Substitutions*

Why we do this (Way of Thinking):
The simplex D is bounded by the three coordinate planes $x = 0$, $y = 0$, $z = 0$ and the oblique plane $x + y + z = 1$. Projecting D onto the xOy-plane by requiring $0 \leq z \leq 1 - x - y$ yields the planar right triangle $D_{xy} = \{0 \leq x \leq 1,\ 0 \leq y \leq 1-x\}$. Because $x + y + z$ enters the integrand symmetrically with unit coefficients, each successive integration in z, y, and x reduces the negative power $(1 + \cdots)^{-k}$ by one degree until the outer integral produces $\ln(1+x)$.

Execution: Step 1 — Cylindrical-fiber decomposition of the tetrahedron D: Requiring $z \geq 0$ and $z \leq 1 - x - y$ implies $1 - x - y \geq 0 \iff x + y \leq 1$. Combined with $x \geq 0$ and $y \geq 0$, the projection onto the xOy-plane is $D_{xy} = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1,\ 0 \leq y \leq 1-x\}$, so

$$D = \{(x,y,z) \in \mathbb{R}^3 : 0 \leq x \leq 1,\ 0 \leq y \leq 1-x,\ 0 \leq z \leq 1-x-y\}.$$

Since $(1+x+y+z)^{-3}$ is continuous on the compact set D (where $1+x+y+z \geq 1$), Fubini's theorem gives

$$I_a = \int_0^1 \left(\int_0^{1-x} \left(\int_0^{1-x-y} \frac{dz}{(1+x+y+z)^3} \right) dy \right) dx.$$

Step 2 — Inner integration with respect to z:

Sub **For fixed $(x,y) \in D_{xy}$, let $u = 1+x+y+z \implies du = dz$. Bounds: $z = 0 \implies u = 1+x+y$; $z = 1-x-y \implies u = 2$.**

$$\int_0^{1-x-y} \frac{dz}{(1+x+y+z)^3} = \int_{1+x+y}^{2} u^{-3}\,du = \left[-\frac{1}{2u^2}\right]_{u=1+x+y}^{u=2}$$
$$= -\frac{1}{8} + \frac{1}{2(1+x+y)^2}.$$

Step 3 — Middle integration with respect to y:

Sub **For fixed $x \in [0,1]$, let $v = 1+x+y \implies dv = dy$. Bounds: $y=0 \implies v = 1+x$; $y = 1-x \implies v = 2$.**

$$\begin{aligned}\int_0^{1-x}\left(\frac{1}{2(1+x+y)^2}-\frac{1}{8}\right)dy &= \frac{1}{2}\int_{1+x}^{2} v^{-2}\,dv - \frac{1}{8}\Big[y\Big]_{y=0}^{y=1-x}\\ &= \frac{1}{2}\left[-\frac{1}{v}\right]_{v=1+x}^{v=2} - \frac{1-x}{8} = -\frac{1}{4}+\frac{1}{2(1+x)}-\frac{1-x}{8}\\ &= \frac{1}{2(1+x)} - \frac{2+(1-x)}{8} = \frac{1}{2(1+x)}-\frac{3-x}{8}.\end{aligned}$$

Step 4 — Outer integration with respect to x:

$$\begin{aligned}I_a &= \int_0^1\left(\frac{1}{2(1+x)}-\frac{3}{8}+\frac{x}{8}\right)dx = \left[\frac{1}{2}\ln(1+x)-\frac{3}{8}x+\frac{x^2}{16}\right]_{x=0}^{x=1}\\ &= \left(\frac{1}{2}\ln 2-\frac{3}{8}+\frac{1}{16}\right)-0 = \frac{1}{2}\ln 2-\frac{6-1}{16} = \frac{1}{2}\ln 2-\frac{5}{16}.\end{aligned}$$

$$\boxed{\iiint_D \frac{dx\,dy\,dz}{(1+x+y+z)^3} = \frac{1}{2}\ln 2-\frac{5}{16}}$$

> ▸ **Common Pitfall:**
> Writing the upper bounds of the three integrals as $\int_0^1\int_0^1\int_0^1$ integrates over the unit cube $[0,1]^3$ rather than the tetrahedron $x+y+z\le 1$. Once a variable is integrated out, it cannot reappear in outer limits: z depends on (x,y), y depends only on x, and the outermost bounds for x are constants.

> ✓ **Sanity Check:**
> On D, $0 \le x+y+z \le 1$, so $1 \le 1+x+y+z \le 2$ and $\frac{1}{8} \le (1+x+y+z)^{-3} \le 1$. Multiplying by the tetrahedron volume $\mathrm{Vol}(D) = \frac{1}{6}$ forces $\frac{1}{48} \approx 0.0208 \le I_a \le \frac{1}{6} \approx 0.1667$; numerically, $\frac{1}{2}\ln 2 - \frac{5}{16} \approx 0.3466 - 0.3125 = 0.0341 \in (0.0208, 0.1667)$.

(b) $f(x,y,z) = y$, **D is bounded by the paraboloid $y = x^2+z^2$ and the plane $y = 4$.** ★★☆☆☆

Method ***Projection onto the xOz-Plane Followed by Polar Coordinates in (x,z)***

> ***Why we do this (Way of Thinking):***
> Here the symmetry axis of the paraboloid $y = x^2+z^2$ is the Oy-axis rather than Oz. Integrating first along the y-direction from $y = x^2+z^2$ up to $y = 4$ projects D onto the circular disk $D_{xz} = \{x^2+z^2 \le 4\}$ in the xOz-plane, after which polar coordinates $(x,z) = (r\cos\theta, r\sin\theta)$ turn the remaining double integral into a one-variable polynomial integral in r.

Execution: Step 1 — Regularity along the Oy-axis and projection onto xOz: A point (x,y,z) lies between the paraboloid $y = x^2+z^2$ and the plane $y = 4$ if and only if $x^2+z^2 \le y \le 4$. For this interval in y to be non-empty, (x,z) must belong to the closed disk

$$D_{xz} = \{(x,z) \in \mathbb{R}^2 : x^2+z^2 \le 4\}.$$

Applying Fubini's theorem by integrating with respect to y first:

$$\begin{aligned}I_b &= \iint_{D_{xz}}\left(\int_{x^2+z^2}^{4} y\,dy\right)dx\,dz = \iint_{D_{xz}}\left[\frac{y^2}{2}\right]_{y=x^2+z^2}^{y=4} dx\,dz\\ &= \frac{1}{2}\iint_{D_{xz}}\left(16-(x^2+z^2)^2\right)dx\,dz.\end{aligned}$$

Step 2 — Polar coordinates in the xOz-plane:

Sub **Set $x = r\cos\theta$ and $z = r\sin\theta$ with $0 \le r \le 2$, $0 \le \theta \le 2\pi$, and area element $dx\,dz = r\,dr\,d\theta$.**

Since $x^2 + z^2 = r^2$, the double integral factors as

$$I_b = \frac{1}{2}\left(\int_0^{2\pi} d\theta\right)\left(\int_0^2 (16 - r^4)\, r\, dr\right) = \frac{1}{2}(2\pi)\int_0^2 (16r - r^5)\, dr$$
$$= \pi\left[8r^2 - \frac{r^6}{6}\right]_{r=0}^{r=2} = \pi\left(8\cdot 4 - \frac{64}{6}\right) = \pi\left(32 - \frac{32}{3}\right) = \frac{64\pi}{3}.$$

$$\iiint_D y\,dx\,dy\,dz = \frac{64\pi}{3}$$

▸ **Common Pitfall:**
Defaulting blindly to standard cylindrical coordinates around Oz ($x = r\cos\theta, y = r\sin\theta$) destroys the rotational symmetry of $y = x^2 + z^2$. Always align polar/cylindrical coordinates with the axis of revolution of the bounding quadric surface (here Oy).

✓ **Sanity Check:**
Verify by cross-sectional slicing perpendicular to Oy: for each $y \in [0,4]$, the slice $D_y = \{(x,z) : x^2+z^2 \le y\}$ is a disk of radius $\sqrt{y}$ and area $\mathrm{Area}(D_y) = \pi y$, so $I_b = \int_0^4 y\ \mathrm{Area}(D_y)\,dy = \pi\int_0^4 y^2\,dy = \pi\left[\frac{y^3}{3}\right]_0^4 = \frac{64\pi}{3}$.

(c) $f(x,y,z) = 2z + x$, $\quad D = \{(x,y,z) \in \mathbb{R}^3 : z \ge 0,\ y \ge x^2 - 1,\ y + z \le 1\}$. ★★★☆☆

Method ***Projection onto the xOy-Plane, Inner Fiber Integration, and Even/Odd Parity in x***

Why we do this (Way of Thinking):
The variable z appears only in the two linear inequalities $0 \le z \le 1 - y$, whereas x enters quadratically through the parabolic cylinder $y \ge x^2 - 1$. Integrating with respect to z first imposes $1 - y \ge 0 \iff y \le 1$, which caps the parabola $y \ge x^2 - 1$ at height $y = 1$ and determines the exact x-range $[-\sqrt{2}, \sqrt{2}]$. Because this projected base D_{xy} is symmetric under $x \mapsto -x$, any odd polynomial term in x drops out automatically at the final stage.

Execution: Step 1 — Projection of D onto the xOy-plane: From $z \ge 0$ and $y + z \le 1$, we have $0 \le z \le 1 - y$, which is non-empty if and only if $y \le 1$. Combining $y \le 1$ with $y \ge x^2 - 1$ requires

$$x^2 - 1 \le y \le 1 \quad\Longrightarrow\quad x^2 - 1 \le 1 \iff x^2 \le 2 \iff -\sqrt{2} \le x \le \sqrt{2}.$$

Hence the projected planar domain is $D_{xy} = \{(x,y) \in \mathbb{R}^2 : -\sqrt{2} \le x \le \sqrt{2},\ x^2 - 1 \le y \le 1\}$ (Figure 6).

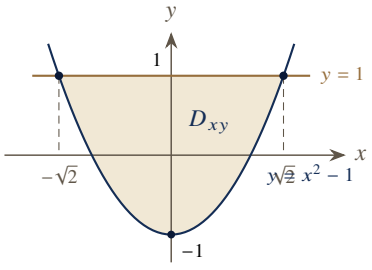


Figure 6: Projected base $D_{xy} = \{-\sqrt{2} \le x \le \sqrt{2},\ x^2 - 1 \le y \le 1\}$ of the solid D.

Step 2 — Inner integration with respect to z: For fixed $(x, y) \in D_{xy}$:

$$\int_0^{1-y} (2z + x)\, dz = \Big[z^2 + xz\Big]_{z=0}^{z=1-y} = (1-y)^2 + x(1-y).$$

Step 3 — Middle integration with respect to y:

Sub **For fixed $x \in [-\sqrt{2}, \sqrt{2}]$, let $u = 1 - y \implies du = -dy$. Bounds: $y = x^2 - 1 \implies u = 1 - (x^2 - 1) = 2 - x^2$; $y = 1 \implies u = 0$.**

$$\int_{x^2-1}^{1} \Big((1-y)^2 + x(1-y)\Big)\, dy = -\int_{2-x^2}^{0} (u^2 + xu)\, du = \int_0^{2-x^2} (u^2 + xu)\, du$$
$$= \left[\frac{u^3}{3} + \frac{xu^2}{2}\right]_{u=0}^{u=2-x^2} = \frac{(2-x^2)^3}{3} + \frac{x(2-x^2)^2}{2}.$$

Step 4 — Outer integration with respect to x via parity: On the symmetric interval $[-\sqrt{2}, \sqrt{2}]$, the second term $g_2(x) = \frac{1}{2}x(2-x^2)^2$ satisfies $g_2(-x) = -g_2(x)$ (odd), so $\int_{-\sqrt{2}}^{\sqrt{2}} g_2(x)\, dx = 0$. The first term $g_1(x) = \frac{1}{3}(2-x^2)^3$ satisfies $g_1(-x) = g_1(x)$ (even), and expanding the cube gives $(2-x^2)^3 = 8 - 12x^2 + 6x^4 - x^6$. Therefore:

$$I_c = \frac{2}{3}\int_0^{\sqrt{2}} (8 - 12x^2 + 6x^4 - x^6)\, dx = \frac{2}{3}\left[8x - 4x^3 + \frac{6}{5}x^5 - \frac{1}{7}x^7\right]_{x=0}^{x=\sqrt{2}}$$
$$= \frac{2}{3}\left(8\sqrt{2} - 4(2\sqrt{2}) + \frac{6}{5}(4\sqrt{2}) - \frac{1}{7}(8\sqrt{2})\right) = \frac{2\sqrt{2}}{3}\left(8 - 8 + \frac{24}{5} - \frac{8}{7}\right)$$
$$= \frac{2\sqrt{2}}{3} \cdot \frac{168 - 40}{35} = \frac{2\sqrt{2}}{3} \cdot \frac{128}{35} = \frac{256\sqrt{2}}{105}.$$

$$\boxed{\iiint_D (2z + x)\, dx\, dy\, dz = \frac{256\sqrt{2}}{105}}$$

> ▸ **Common Pitfall:**
> Setting the x-bounds to $[-1, 1]$ (the x-intercepts of $y = x^2 - 1$ with $y = 0$) truncates the solid: the plane $y + z = 1$ meets $z = 0$ along the line $y = 1$, not $y = 0$, so $x^2 - 1 \le 1$ extends all the way to $x = \pm\sqrt{2}$.

> ✓ **Sanity Check:**
> Cross-check by horizontal slicing at fixed $z \in [0, 2]$: by reflection symmetry $x \mapsto -x$, $\iiint_D x\, dV = 0$, while each slice $D_z = \{|x| \le \sqrt{2 - z},\ x^2 - 1 \le y \le 1 - z\}$ has area $\int_{-\sqrt{2-z}}^{\sqrt{2-z}} (2 - z - x^2)\, dx = \frac{4}{3}(2-z)^{3/2}$. Setting $t = \sqrt{2-z}$ ($z = 2 - t^2$, $dz = -2t\, dt$) yields $\int_0^2 2z \cdot \frac{4}{3}(2-z)^{3/2}\, dz = \frac{16}{3}\int_0^{\sqrt{2}} (2t^4 - t^6)\, dt = \frac{16\sqrt{2}}{3}\left(\frac{8}{5} - \frac{8}{7}\right) = \frac{256\sqrt{2}}{105}$.

Exercise 6

Compute $\iiint_D f(x, y, z)\, dx\, dy\, dz$ *in each of the following cases:*

(a) $f(x,y,z)=\dfrac{z}{\sqrt{x^2+y^2+z^2}}$, $D=\{(x,y,z)\in\mathbb{R}^3 : 1\le x^2+y^2+z^2\le 4,\ z\ge\sqrt{x^2+y^2}\}$. ★★☆☆☆

Method ***Spherical Coordinates on a Conical Spherical Shell***

> ***Why we do this (Way of Thinking):***
> Both the radial shell $1\le x^2+y^2+z^2\le 4$ and the upper half-cone $z\ge\sqrt{x^2+y^2}$ are constant-coordinate surfaces in spherical coordinates (ρ,θ,φ), where $\rho=\sqrt{x^2+y^2+z^2}$ and $\varphi\in[0,\pi]$ is the colatitude measured from the positive Oz-axis. Moreover, $\frac{z}{\sqrt{x^2+y^2+z^2}}=\cos\varphi$, so D becomes a rectangular box in (ρ,θ,φ) with a completely separable integrand.

Execution: Step 1 — Spherical parameterization of D:

Sub **Pass to spherical coordinates $x=\rho\sin\varphi\cos\theta$, $y=\rho\sin\varphi\sin\theta$, $z=\rho\cos\varphi$ ($\rho>0$, $\theta\in[0,2\pi[$, $\varphi\in[0,\pi[$), with volume element $dx\,dy\,dz=\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$.**

Translating the inequalities defining D (Figure 7):

- $1\le x^2+y^2+z^2\le 4 \iff 1\le\rho^2\le 4 \iff 1\le\rho\le 2$ (since $\rho>0$);
- $\sqrt{x^2+y^2}=\rho\sin\varphi$ (as $\sin\varphi\ge 0$ for $\varphi\in[0,\pi]$), so $z\ge\sqrt{x^2+y^2} \iff \rho\cos\varphi\ge\rho\sin\varphi \iff \cos\varphi\ge\sin\varphi \iff 0\le\varphi\le\dfrac{\pi}{4}$;
- θ is unconstrained: $0\le\theta\le 2\pi$.

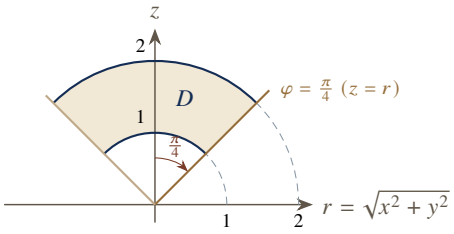


Figure 7: Meridian cross-section of the conical spherical shell $D=\{1\le\rho\le 2,\ 0\le\theta\le 2\pi,\ 0\le\varphi\le\frac{\pi}{4}\}$.

Step 2 — Separable evaluation of the triple integral: Multiplying $f(\rho\sin\varphi\cos\theta,\rho\sin\varphi\sin\theta,\rho\cos\varphi)=\frac{\rho\cos\varphi}{\rho}=\cos\varphi$ by the spherical Jacobian $\rho^2\sin\varphi$:

$$\begin{aligned} I_a &= \iiint_{[1,2]\times[0,2\pi]\times[0,\pi/4]} \rho^2\sin\varphi\cos\varphi\,d\rho\,d\theta\,d\varphi \\ &= \left(\int_0^{2\pi} d\theta\right)\left(\int_1^2 \rho^2\,d\rho\right)\left(\int_0^{\pi/4}\sin\varphi\cos\varphi\,d\varphi\right) \\ &= (2\pi)\cdot\left[\frac{\rho^3}{3}\right]_{\rho=1}^{\rho=2}\cdot\left[\frac{\sin^2\varphi}{2}\right]_{\varphi=0}^{\varphi=\pi/4} \\ &= 2\pi\cdot\frac{8-1}{3}\cdot\frac{\left(\frac{\sqrt{2}}{2}\right)^2-0}{2} = 2\pi\cdot\frac{7}{3}\cdot\frac{1}{4}=\frac{7\pi}{6}. \end{aligned}$$

$$\boxed{\iiint_D \frac{z}{\sqrt{x^2+y^2+z^2}}\,dx\,dy\,dz=\frac{7\pi}{6}}$$

> ► **Common Pitfall:**
> Confusing the colatitude φ (angle from $+Oz$, where the cone $z=\sqrt{x^2+y^2}$ is $\varphi=\frac{\pi}{4}$ and the $+Oz$-axis is $\varphi=0$) with geographical latitude (angle from the xOy-plane) would wrongly replace $\varphi\in\left[0,\frac{\pi}{4}\right]$ by $\left[\frac{\pi}{4},\frac{\pi}{2}\right]$ or swap $\sin\varphi$ and $\cos\varphi$ in the Jacobian $\rho^2\sin\varphi$.

> ✓ **Sanity Check:**
> On D, $\varphi \in \left[0, \frac{\pi}{4}\right] \implies \frac{\sqrt{2}}{2} \le \cos\varphi \le 1$, and $\mathrm{Vol}(D) = 2\pi \cdot \frac{7}{3} \cdot \left(1 - \frac{\sqrt{2}}{2}\right) \approx 4.292$. Hence $\frac{\sqrt{2}}{2}\,\mathrm{Vol}(D) \approx 3.035 \le I_a \le 4.292$, which agrees with $\frac{7\pi}{6} \approx 3.665$.

(b) $f(x,y,z) = z\,e^{x^2+y^2}$, $\quad D = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le 1,\ z \ge 0\right\}$. ★★☆☆☆

Method ***Cylindrical Coordinates Followed by Radial Substitution and Integration by Parts***

> ***Why we do this (Way of Thinking):***
> Although D is the upper unit hemisphere, spherical coordinates would turn $e^{x^2+y^2}$ into $e^{\rho^2\sin^2\varphi}$, coupling ρ and φ non-elementarily. By contrast, cylindrical coordinates (r,θ,z) convert $e^{x^2+y^2}$ into e^{r^2} (independent of z and θ), while the factor z integrates over $z \in \left[0, \sqrt{1-r^2}\right]$ to $\frac{1}{2}(1-r^2)$, removing the square root completely and leaving the substitution $u = r^2$ paired with $r\,dr$.

Execution: Step 1 — Cylindrical coordinates on the upper unit hemisphere D:

Sub **Set $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, with $dx\,dy\,dz = r\,dr\,d\theta\,dz$. On D, $0 \le \theta \le 2\pi$, $0 \le r \le 1$, and $0 \le z \le \sqrt{1-r^2}$.**

Applying Fubini's theorem by integrating with respect to z first:

$$I_b = \int_0^{2\pi} d\theta \int_0^1 e^{r^2} r \left(\int_0^{\sqrt{1-r^2}} z\,dz \right) dr$$
$$= 2\pi \int_0^1 e^{r^2} r \left[\frac{z^2}{2} \right]_{z=0}^{z=\sqrt{1-r^2}} dr = \pi \int_0^1 (1-r^2)\, e^{r^2} r\,dr.$$

Step 2 — Radial substitution and integration by parts:

Sub **Let $u = r^2 \implies du = 2r\,dr \implies r\,dr = \frac{1}{2}\,du$. Bounds: $r = 0 \implies u = 0$; $r = 1 \implies u = 1$.**

$$I_b = \frac{\pi}{2} \int_0^1 (1-u) e^u\,du.$$

Integrate by parts with $U(u) = 1-u$ $(U'(u) = -1)$ and $V'(u) = e^u$ $(V(u) = e^u)$:

$$\int_0^1 (1-u)e^u\,du = \Big[(1-u)e^u\Big]_{u=0}^{u=1} - \int_0^1 (-1)e^u\,du$$
$$= \left(0 - 1\cdot e^0\right) + \Big[e^u\Big]_{u=0}^{u=1} = -1 + (e-1) = e-2.$$

Multiplying by $\frac{\pi}{2}$ yields

$$\boxed{\iiint_D z\,e^{x^2+y^2}\,dx\,dy\,dz = \frac{\pi(e-2)}{2}}$$

> ► **Common Pitfall:**
> Choosing spherical coordinates simply because the boundary $x^2+y^2+z^2 = 1$ is a sphere is a classic strategic trap: always inspect the *integrand* $(e^{x^2+y^2} = e^{r^2})$ alongside the domain before choosing between cylindrical and spherical coordinates.

> ✓ **Sanity Check:**
> Differentiate $(2-u)e^u$ to check the primitive: $\frac{d}{du}\left[(2-u)e^u\right] = -e^u + (2-u)e^u = (1-u)e^u$, and $\left[(2-u)e^u\right]_0^1 = e - 2 > 0$ (since $e \approx 2.7183 > 2$). Also, on D, $z \le ze^{r^2} \le ez$, and $\iiint_D z\,dV = \frac{\pi}{4}$, so $\frac{\pi}{4} \approx 0.785 \le I_b \le \frac{\pi e}{4} \approx 2.135$; indeed $\frac{\pi(e-2)}{2} \approx 1.128$.

(c) $f(x,y,z) = \sqrt{x^2+y^2+(z-1)^2}$, $\quad D = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le 4\}$. ★★★★☆

Method ***Spherical Coordinates with Colatitude-First Substitution and Piecewise Absolute-Value Splitting***

> ***Why we do this (Way of Thinking):***
> Geometrically, $f(x,y,z)$ is the Euclidean distance from (x,y,z) to the off-center point $A = (0,0,1)$, which lies *strictly inside* the ball D of radius 2. Passing to spherical coordinates centered at the origin turns $x^2+y^2+(z-1)^2$ into $\rho^2 - 2\rho\cos\varphi + 1$. Integrating with respect to $\varphi \in [0,\pi]$ *before* ρ allows $\sin\varphi\,d\varphi$ from the spherical Jacobian $\rho^2\sin\varphi$ to supply the exact differential of $-2\rho\cos\varphi$. At the lower bound $\varphi = 0$, however, $\sqrt{(\rho-1)^2} = |\rho-1|$, forcing us to split the radial integral at $\rho = 1$.

Execution: Step 1 — Spherical setup on the ball of radius 2:

Sub **Set $x = \rho\sin\varphi\cos\theta$, $y = \rho\sin\varphi\sin\theta$, $z = \rho\cos\varphi$ on $\Delta = [0,2]\times[0,2\pi]\times[0,\pi]$, $dx\,dy\,dz = \rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$.**

Expanding the squared distance inside the radical:

$$x^2+y^2+(z-1)^2 = (x^2+y^2+z^2) - 2z + 1 = \rho^2 - 2\rho\cos\varphi + 1.$$

Integrating $\theta \in [0,2\pi]$ immediately and placing the φ-integral inside the ρ-integral:

$$I_c = 2\pi\int_0^2 \rho^2\left(\int_0^\pi \sqrt{\rho^2-2\rho\cos\varphi+1}\,\sin\varphi\,d\varphi\right)d\rho.$$

Step 2 — Inner φ-integral and absolute-value evaluation: Fix $\rho \in (0,2]$.

Sub **Let $u = \rho^2 - 2\rho\cos\varphi + 1 \implies du = 2\rho\sin\varphi\,d\varphi \implies \sin\varphi\,d\varphi = \frac{du}{2\rho}$. Bounds: $\varphi = 0 \implies u = \rho^2 - 2\rho + 1 = (\rho-1)^2$; $\varphi = \pi \implies u = \rho^2 + 2\rho + 1 = (\rho+1)^2$.**

Since $\left((\rho\pm1)^2\right)^{3/2} = |\rho\pm1|^3$ and $\rho + 1 > 0$ for $\rho > 0$, we obtain

$$J(\rho) = \int_0^\pi \sqrt{\rho^2-2\rho\cos\varphi+1}\,\sin\varphi\,d\varphi = \frac{1}{2\rho}\int_{(\rho-1)^2}^{(\rho+1)^2} u^{1/2}\,du$$
$$= \frac{1}{2\rho}\left[\frac{2}{3}u^{3/2}\right]_{u=(\rho-1)^2}^{u=(\rho+1)^2} = \frac{1}{3\rho}\Big((\rho+1)^3 - |\rho-1|^3\Big).$$

Step 3 — Piecewise simplification on $[0,1]$ and $[1,2]$:

- **On $\rho \in [0,1]$:** $|\rho-1| = 1-\rho$, so

$$(\rho+1)^3 - (1-\rho)^3 = (\rho^3+3\rho^2+3\rho+1) - (1-3\rho+3\rho^2-\rho^3) = 2\rho^3 + 6\rho = 2\rho(\rho^2+3),$$

  and therefore $\rho^2 J(\rho) = \frac{\rho^2}{3\rho}\cdot 2\rho(\rho^2+3) = \frac{2}{3}(\rho^4+3\rho^2)$.

- **On $\rho \in [1,2]$:** $|\rho-1| = \rho-1$, so

$$(\rho+1)^3 - (\rho-1)^3 = (\rho^3+3\rho^2+3\rho+1) - (\rho^3-3\rho^2+3\rho-1) = 6\rho^2 + 2 = 2(3\rho^2+1),$$

  and therefore $\rho^2 J(\rho) = \frac{\rho^2}{3\rho}\cdot 2(3\rho^2+1) = \frac{2}{3}(3\rho^3+\rho)$.

Step 4 — Radial integration across $[0,1]\cup[1,2]$:

$$\begin{aligned} I_c &= 2\pi\cdot\frac{2}{3}\left(\int_0^1(\rho^4+3\rho^2)\,d\rho+\int_1^2(3\rho^3+\rho)\,d\rho\right)\\ &=\frac{4\pi}{3}\left(\left[\frac{\rho^5}{5}+\rho^3\right]_{\rho=0}^{\rho=1}+\left[\frac{3\rho^4}{4}+\frac{\rho^2}{2}\right]_{\rho=1}^{\rho=2}\right)\\ &=\frac{4\pi}{3}\left[\left(\frac{1}{5}+1\right)+\left(\left(\frac{3\cdot16}{4}+\frac{4}{2}\right)-\left(\frac{3}{4}+\frac{1}{2}\right)\right)\right]\\ &=\frac{4\pi}{3}\left(\frac{6}{5}+14-\frac{5}{4}\right)=\frac{4\pi}{3}\left(\frac{6}{5}+\frac{51}{4}\right)=\frac{4\pi}{3}\cdot\frac{24+255}{20}=\frac{4\pi}{3}\cdot\frac{279}{20}=\frac{93\pi}{5}. \end{aligned}$$

$$\boxed{\iiint_D\sqrt{x^2+y^2+(z-1)^2}\,dx\,dy\,dz=\frac{93\pi}{5}}$$

> ▸ **Common Pitfall:**
> Writing $\big((\rho-1)^2\big)^{3/2}=(\rho-1)^3$ across the entire interval $\rho\in[0,2]$ without absolute value bars treats $|\rho-1|$ as negative on $[0,1)$, yielding $\frac{4\pi}{3}\int_0^2(3\rho^3+\rho)\,d\rho=\frac{4\pi}{3}(12+2)=\frac{56\pi}{3}\approx58.643$ instead of the true value $\frac{93\pi}{5}\approx58.434$. Always check whether $\rho-a$ changes sign inside the radial integration range.

> ✓ **Sanity Check:**
> Notice the physical continuity at the shell $\rho=1$: from the left ($\rho\to1^-$), $\rho^2J(\rho)=\frac{2}{3}(1+3)=\frac{8}{3}$; from the right ($\rho\to1^+$), $\rho^2J(\rho)=\frac{2}{3}(3+1)=\frac{8}{3}$. Moreover, the average distance from $(0,0,1)$ to points in D is $\frac{I_c}{\mathrm{Vol}(D)}=\frac{93\pi/5}{32\pi/3}=\frac{279}{160}=1.74375$, which lies strictly between 0 and the maximum distance 3 (attained at $(0,0,-2)$).

(d) $f(x,y,z)=xyz+\dfrac{x+y+z}{xyz}$, $\quad D=\{(x,y,z)\in\mathbb{R}^3:1\le xy\le4,\ 1\le yz\le9,\ 1\le xz\le4\}$. ★★★★★

Method ***Two-Octant Component Analysis, Central-Inversion Parity, and Pairwise-Product Change of Variables***

> ***Why we do this (Way of Thinking):***
> Before setting $u=xy$, $v=yz$, $w=xz$, inspect the sign constraints carefully: $xy\ge1>0$, $yz\ge1>0$, and $xz\ge1>0$ require x, y, and z to share the *same* sign, which holds in *two* disjoint octants: $(0,+\infty)^3$ (all positive) and $(-\infty,0)^3$ (all negative). Thus $\psi(x,y,z)=(xy,yz,xz)$ is two-to-one on D and becomes bijective only after decomposing $D=D_+\sqcup D_-$ with $D_-=-D_+$. Under central inversion $(x,y,z)\mapsto(-x,-y,-z)$, the odd term xyz cancels completely between D_+ and D_-, whereas the even term $\frac{x+y+z}{xyz}$ doubles on D_+, where (u,v,w) straightens D_+ into $[1,4]\times[1,9]\times[1,4]$.

Execution: Step 1 — Decomposition of D into two disjoint octant components: On D, $xy\ge1$, $yz\ge1$, and $xz\ge1$ prevent any coordinate from vanishing. If $x>0$, then $y=\frac{xy}{x}>0$ and $z=\frac{xz}{x}>0$; if $x<0$, then $y=\frac{xy}{x}<0$ and $z=\frac{xz}{x}<0$. Hence D is the disjoint union $D=D_+\sqcup D_-$, where

$$D_+=D\cap(0,+\infty)^3,\qquad D_-=D\cap(-\infty,0)^3=T(D_+),$$

and $T(x,y,z)=(-x,-y,-z)$ is the central inversion across the origin (with constant Jacobian $|\det J_T|=|(-1)^3|=1$).

Step 2 — Parity reduction under central inversion T: Decompose $f=f_1+f_2$ with $f_1(x,y,z)=xyz$ and $f_2(x,y,z)=\dfrac{x+y+z}{xyz}$:

- $f_1(-x,-y,-z) = (-x)(-y)(-z) = -xyz = -f_1(x,y,z)$ (odd under T), so applying the change of variables $(x,y,z) = T(\xi,\eta,\zeta)$ on D_- gives $\iiint_{D_-} f_1\,dV = -\iiint_{D_+} f_1\,dV$, and hence

$$\iiint_D xyz\,dx\,dy\,dz = \iiint_{D_+} xyz\,dV + \iiint_{D_-} xyz\,dV = 0.$$

- $f_2(-x,-y,-z) = \dfrac{-(x+y+z)}{-xyz} = \dfrac{x+y+z}{xyz} = f_2(x,y,z)$ (even under T), so $\iiint_{D_-} f_2\,dV = \iiint_{D_+} f_2\,dV$, and hence

$$I_d = \iiint_D f(x,y,z)\,dx\,dy\,dz = 2\iiint_{D_+} \frac{x+y+z}{xyz}\,dx\,dy\,dz.$$

Step 3 — Bijectivity and Jacobian of $(u,v,w) = (xy, yz, xz)$ on D_+:

Sub **On $D_+ \subset (0,+\infty)^3$, define $\psi(x,y,z) = (u,v,w)$ by $u = xy$, $v = yz$, $w = xz$.**

For $(x,y,z) \in D_+$, $uvw = (xy)(yz)(xz) = (xyz)^2$ with $xyz > 0$, so $xyz = \sqrt{uvw}$, yielding the unique C^1 inverse $\phi : \Delta \to D_+$ on $\Delta = [1,4]\times[1,9]\times[1,4]$ given by

$$x = \frac{xyz}{yz} = \sqrt{\frac{uw}{v}}, \qquad y = \frac{xyz}{xz} = \sqrt{\frac{uv}{w}}, \qquad z = \frac{xyz}{xy} = \sqrt{\frac{vw}{u}}.$$

Computing the forward Jacobian determinant of ψ by cofactor expansion along the first row:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} y & x & 0 \\ 0 & z & y \\ z & 0 & x \end{vmatrix} = y(xz-0) - x(0-yz) + 0 = xyz + xyz = 2xyz = 2\sqrt{uvw} > 0,$$

so by the inverse function theorem, $\left|J_\phi(u,v,w)\right| = \dfrac{1}{2\sqrt{uvw}} = \dfrac{1}{2}u^{-1/2}v^{-1/2}w^{-1/2}$.

Step 4 — Separable evaluation on $\Delta = [1,4]\times[1,9]\times[1,4]$: Rewrite f_2 in terms of (u,v,w):

$$\frac{x+y+z}{xyz} = \frac{1}{yz} + \frac{1}{xz} + \frac{1}{xy} = \frac{1}{v} + \frac{1}{w} + \frac{1}{u}.$$

Substituting into $I_d = 2\iiint_{D_+} f_2\,dx\,dy\,dz$ cancels the factor of 2:

$$\begin{aligned} I_d &= 2\iiint_{[1,4]\times[1,9]\times[1,4]} \left(\frac{1}{u}+\frac{1}{v}+\frac{1}{w}\right)\cdot\frac{1}{2}u^{-1/2}v^{-1/2}w^{-1/2}\,du\,dv\,dw \\ &= \iiint_{[1,4]\times[1,9]\times[1,4]} \left(u^{-3/2}v^{-1/2}w^{-1/2} + u^{-1/2}v^{-3/2}w^{-1/2} + u^{-1/2}v^{-1/2}w^{-3/2}\right)du\,dv\,dw. \end{aligned}$$

Evaluate the four elementary one-dimensional power integrals:

$$\int_1^4 t^{-1/2}\,dt = \left[2t^{1/2}\right]_1^4 = 2(2-1) = 2, \qquad \int_1^9 t^{-1/2}\,dt = \left[2t^{1/2}\right]_1^9 = 2(3-1) = 4,$$
$$\int_1^4 t^{-3/2}\,dt = \left[-2t^{-1/2}\right]_1^4 = -2\left(\frac{1}{2}-1\right) = 1, \qquad \int_1^9 t^{-3/2}\,dt = \left[-2t^{-1/2}\right]_1^9 = -2\left(\frac{1}{3}-1\right) = \frac{4}{3}.$$

Summing the three factored triple integrals:

$$I_d = (1)(4)(2) + (2)\left(\frac{4}{3}\right)(2) + (2)(4)(1) = 8 + \frac{16}{3} + 8 = \frac{64}{3}.$$

$$\boxed{\iiint_D \left(xyz + \frac{x+y+z}{xyz}\right)dx\,dy\,dz = \frac{64}{3}}$$

> ▸ **Common Pitfall:**
> Overlooking the negative-octant component $D_- \subset (-\infty, 0)^3$ (assuming $xy, yz, xz > 0 \implies x, y, z > 0$) produces two simultaneous errors: it keeps the non-zero positive value $\iiint_{D_+} xyz\,dV = \frac{1}{2}(4-1)(9-1)(4-1) = 36$ (which actually cancels against -36 from D_-) and halves the true contribution of $\frac{x+y+z}{xyz}$ to $\frac{32}{3}$, giving $36 + \frac{32}{3} = \frac{140}{3}$ instead of $\frac{64}{3}$.

> ✓ **Sanity Check:**
> Verify $\mathrm{Vol}(D_+)$ and bounds on f_2: $\mathrm{Vol}(D_+) = \frac{1}{2}\left(\int_1^4 u^{-1/2}\,du\right)\left(\int_1^9 v^{-1/2}\,dv\right)\left(\int_1^4 w^{-1/2}\,dw\right) = \frac{1}{2}(2)(4)(2) = 8$, so $\mathrm{Vol}(D) = 2\,\mathrm{Vol}(D_+) = 16$. On D_+, $\frac{1}{4} + \frac{1}{9} + \frac{1}{4} = \frac{11}{18} \le \frac{1}{u} + \frac{1}{v} + \frac{1}{w} \le 1 + 1 + 1 = 3$, forcing $16 \cdot \frac{11}{18} \approx 9.78 \le I_d \le 16 \cdot 3 = 48$; indeed $\frac{64}{3} \approx 21.33 \in (9.78, 48)$.

Exercise 7

Determine the volumes of the following solids:

(a) $V_1 = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \le R^2,\ x^2 + y^2 - Rx \le 0\right\}, \quad R > 0.$ ★★★★☆

Method ***First-Octant Symmetry Reduction and Cylindrical Coordinates (Viviani's Solid)***

> ***Why we do this (Way of Thinking):***
> Completing the square in $x^2 + y^2 - Rx \le 0$ gives $\left(x - \frac{R}{2}\right)^2 + y^2 \le \left(\frac{R}{2}\right)^2$, an off-center vertical circular cylinder of diameter R passing through the origin and tangent to the Oz-axis, drilled through the sphere of radius R. In cylindrical coordinates, this base disk is $0 \le r \le R\cos\theta$ for $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Restricting to the first octant $\theta \in \left[0, \frac{\pi}{2}\right]$, $z \ge 0$ (multiplying by 4 by reflection symmetry across $z = 0$ and $y = 0$) guarantees $\sin\theta \ge 0$, so $\left(R^2\sin^2\theta\right)^{3/2} = R^3\sin^3\theta$ without absolute-value splitting.

Execution: Step 1 — Four-fold reflection symmetry and cylindrical coordinates: Both inequalities $x^2 + y^2 + z^2 \le R^2$ and $x^2 + y^2 - Rx \le 0$ are invariant under $(x, y, z) \mapsto (x, y, -z)$ and $(x, y, z) \mapsto (x, -y, z)$. Therefore $\mathrm{Vol}(V_1) = 4\,\mathrm{Vol}(V_{1,+})$, where $V_{1,+} = \left\{(x, y, z) \in V_1 : y \ge 0,\ z \ge 0\right\}$ (Figure 8).

Sub **Pass to cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, $dx\,dy\,dz = r\,dr\,d\theta\,dz$.**

On $V_{1,+}$:

- $x^2 + y^2 - Rx \le 0 \implies Rx \ge x^2 + y^2 \ge 0 \implies x \ge 0$; combined with $y \ge 0$, $\theta \in \left[0, \frac{\pi}{2}\right]$, and $r^2 - Rr\cos\theta \le 0 \iff 0 \le r \le R\cos\theta$;
- $0 \le z \le \sqrt{R^2 - r^2}$.

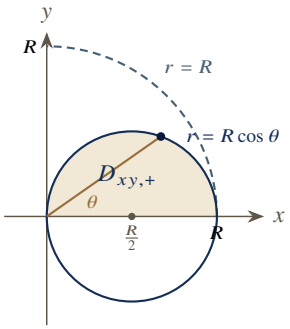


Figure 8: First-quadrant base half-disk $D_{xy,+} = \left\{0 \le \theta \le \frac{\pi}{2},\ 0 \le r \le R\cos\theta\right\}$ inside the sphere cross-section $r \le R$.

Step 2 — Inner z- and middle r-integrations:

$$\mathrm{Vol}(V_1) = 4\int_0^{\pi/2}\left(\int_0^{R\cos\theta}\left(\int_0^{\sqrt{R^2-r^2}} dz\right) r\,dr\right) d\theta = 4\int_0^{\pi/2}\left(\int_0^{R\cos\theta}\sqrt{R^2-r^2}\,r\,dr\right) d\theta.$$

Sub **For fixed $\theta \in \left[0, \frac{\pi}{2}\right]$, let $u = R^2 - r^2 \implies \boldsymbol{du = -2r\,dr} \implies r\,dr = -\frac{1}{2}\,du$. Bounds: $r = 0 \implies u = R^2$; $r = R\cos\theta \implies u = R^2(1-\cos^2\theta) = R^2\sin^2\theta$.**

Since $\sin\theta \geq 0$ on $\left[0, \frac{\pi}{2}\right]$, $\left(R^2\sin^2\theta\right)^{3/2} = R^3\sin^3\theta$:

$$\int_0^{R\cos\theta}\sqrt{R^2-r^2}\,r\,dr = -\frac{1}{2}\int_{R^2}^{R^2\sin^2\theta} u^{1/2}\,du = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_{u=R^2\sin^2\theta}^{u=R^2} = \frac{R^3}{3}\left(1-\sin^3\theta\right).$$

Step 3 — Outer θ-integration: Rewrite $\sin^3\theta = (1-\cos^2\theta)\sin\theta$:

Sub **Let $w = \cos\theta \implies \boldsymbol{dw = -\sin\theta\,d\theta}$. Bounds: $\theta = 0 \implies w = 1$; $\theta = \frac{\pi}{2} \implies w = 0$.**

$$\int_0^{\pi/2}\sin^3\theta\,d\theta = -\int_1^0 (1-w^2)\,dw = \int_0^1 (1-w^2)\,dw = \left[w - \frac{w^3}{3}\right]_{w=0}^{w=1} = 1 - \frac{1}{3} = \frac{2}{3}.$$

Substituting back gives

$$\mathrm{Vol}(V_1) = 4\cdot\frac{R^3}{3}\int_0^{\pi/2}\left(1-\sin^3\theta\right)d\theta = \frac{4R^3}{3}\left(\frac{\pi}{2} - \frac{2}{3}\right) = \frac{4R^3}{3}\cdot\frac{3\pi-4}{6} = \frac{2}{9}(3\pi-4)R^3.$$

$$\boxed{\mathrm{Vol}(V_1) = \frac{2}{9}(3\pi-4)R^3}$$

> ► **Common Pitfall:**
> Integrating over the full interval $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ without restricting to $\left[0, \frac{\pi}{2}\right]$ and writing $\left(R^2\sin^2\theta\right)^{3/2} = R^3\sin^3\theta$ makes $\sin^3\theta$ odd on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, falsely cancelling the $-\frac{2}{3}$ term and producing $\frac{2\pi}{3}R^3$ (half the full sphere) instead of $\frac{2}{9}(3\pi-4)R^3$, because $\left(\sin^2\theta\right)^{3/2} = |\sin\theta|^3$, not $\sin^3\theta$.

> ✓ **Sanity Check:**
> Since V_1 lies strictly inside the right half-ball $\{x \geq 0,\ x^2+y^2+z^2 \leq R^2\}$ of volume $\frac{2\pi}{3}R^3 \approx 2.094\,R^3$, its volume must satisfy $0 < \mathrm{Vol}(V_1) < \frac{2\pi}{3}R^3$; indeed $\frac{2}{9}(3\pi-4)R^3 \approx 1.205\,R^3 \in (0, 2.094\,R^3)$ with correct cubic dimension $[R^3]$.

(b) $V_2 = \left\{(x,y,z) \in \mathbb{R}^3 : x^2 + 2y^2 + 6z^2 + 2xy + 2xz \leq 1\right\}$. ★★★☆☆

Method *Gauss Completion of Squares and Linear Change of Variables onto the Unit Ball*

> ***Why we do this (Way of Thinking):***
> A positive-definite quadratic inequality $Q(x,y,z) \leq 1$ with cross-terms ($2xy$ and $2xz$) defines a tilted ellipsoid in $\mathbb{R}^3$. By completing the square one variable at a time (Gauss's reduction of quadratic forms), we express $Q(x,y,z)$ as a sum of three independent squares $u^2+v^2+w^2 \leq 1$. Because (u,v,w) is constructed sequentially, its forward Jacobian matrix is upper triangular, so its determinant is the product of the diagonal entries and $\mathrm{Vol}(V_2)$ follows immediately from the unit-ball volume $\frac{4\pi}{3}$.

Execution: Step 1 — Sequential completion of squares in $Q(x,y,z)$: Group all terms containing x first, then complete the remaining quadratic in (y,z):

$$\begin{aligned} Q(x,y,z) &= x^2 + 2x(y+z) + 2y^2 + 6z^2 \\ &= (x+y+z)^2 - (y+z)^2 + 2y^2 + 6z^2 \\ &= (x+y+z)^2 - (y^2 + 2yz + z^2) + 2y^2 + 6z^2 \\ &= (x+y+z)^2 + y^2 - 2yz + 5z^2 \\ &= (x+y+z)^2 + (y-z)^2 - z^2 + 5z^2 \\ &= (x+y+z)^2 + (y-z)^2 + (2z)^2. \end{aligned}$$

Step 2 — Linear coordinate transformation onto the unit ball B_1:

Sub **Define $\psi(x,y,z) = (u,v,w)$ by $u = x+y+z$, $v = y-z$, $w = 2z$, so $(x,y,z) \in V_2 \iff u^2+v^2+w^2 \leq 1$.**

The forward Jacobian matrix of ψ is upper triangular:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{vmatrix} = 1 \cdot 1 \cdot 2 = 2 \neq 0,$$

so $\psi : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear isomorphism mapping V_2 bijectively onto the closed unit ball $B_1 = \{(u,v,w) \in \mathbb{R}^3 : u^2+v^2+w^2 \leq 1\}$, and its inverse $\phi = \psi^{-1}$ has constant Jacobian magnitude $\left|J_\phi(u,v,w)\right| = \dfrac{1}{2}$.

Step 3 — Volume computation: Passing to spherical coordinates (ρ,θ,φ) on B_1 ($\rho \in [0,1]$, $\theta \in [0,2\pi]$, $\varphi \in [0,\pi]$):

$$\begin{aligned} \mathrm{Vol}(V_2) &= \iiint_{V_2} dx\,dy\,dz = \iiint_{B_1} \frac{1}{2}\,du\,dv\,dw \\ &= \frac{1}{2}\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin\varphi\,d\varphi\right)\left(\int_0^{1} \rho^2\,d\rho\right) = \frac{1}{2}\cdot(2\pi)\cdot(2)\cdot\frac{1}{3} = \frac{2\pi}{3}. \end{aligned}$$

$$\boxed{\mathrm{Vol}(V_2) = \frac{2\pi}{3}}$$

► **Common Pitfall:**
Multiplying $\mathrm{Vol}(B_1) = \frac{4\pi}{3}$ by the forward determinant $\left|\frac{\partial(u,v,w)}{\partial(x,y,z)}\right| = 2$ instead of $\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \frac{1}{2}$ overestimates the volume by a factor of 4 (giving $\frac{8\pi}{3}$). Remember that $du\,dv\,dw = 2\,dx\,dy\,dz \iff dx\,dy\,dz = \frac{1}{2}\,du\,dv\,dw$.

✓ **Sanity Check:**
Write $Q(x,y,z) = X^{\mathsf{T}}AX$ with symmetric matrix $A = \left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 6 \end{smallmatrix}\right)$: its leading principal minors are $1 > 0$, $2 - 1 = 1 > 0$, and $\det A = 1(12) - 1(6) + 1(-2) = 4 > 0$, confirming positive definiteness and $\mathrm{Vol}(V_2) = \frac{4\pi/3}{\sqrt{\det A}} = \frac{4\pi/3}{2} = \frac{2\pi}{3}$.

(c) $V_3 = \left\{(x,y,z) \in \mathbb{R}^3 : \left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} + \left(\frac{z}{c}\right)^{2/3} \le 1\right\}, \quad a,b,c > 0.$ ★★★★☆

Method ***Eight-Octant Symmetry, Cubic Power-Rescaling Map, and Spherical Coordinates***

> ***Why we do this (Way of Thinking):***
> Since $t^{2/3} = \left(\sqrt[3]{t}\right)^2 = |t|^{2/3}$ is even in each coordinate, the astroidal solid V_3 consists of 8 congruent octants. Working strictly inside the open first octant $(0,+\infty)^3$ avoids zero-derivative singularities of $t \mapsto t^3$ on the coordinate planes and lets the cubic substitution $x = au^3$, $y = bv^3$, $z = cw^3$ turn $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} + \left(\frac{z}{c}\right)^{2/3} \le 1$ into the first-octant unit ball $u^2 + v^2 + w^2 \le 1$, which factors completely in spherical coordinates.

Execution: Step 1 — Eight-octant symmetry and cubic rescaling onto the first-octant unit ball: Because V_3 is invariant under $(x,y,z) \mapsto (\pm x, \pm y, \pm z)$ and the three coordinate planes have zero three-dimensional volume, $\mathrm{Vol}(V_3) = 8\ \mathrm{Vol}(V_{3,+})$, where $V_{3,+} = V_3 \cap (0,+\infty)^3$.

Sub **Stage 1: Set $x = au^3$, $y = bv^3$, $z = cw^3$ for $(u,v,w) \in (0,+\infty)^3$.**

Then $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} + \left(\frac{z}{c}\right)^{2/3} = u^2+v^2+w^2$, mapping the first-octant unit ball $B_{1,+} = \left\{(u,v,w) \in (0,+\infty)^3 : u^2 + v^2 + w^2 \le 1\right\}$ bijectively onto $V_{3,+}$ with diagonal Jacobian

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = (3au^2)(3bv^2)(3cw^2) = 27abc\,u^2v^2w^2 > 0.$$

Hence

$$\mathrm{Vol}(V_3) = 8 \iiint_{B_{1,+}} 27abc\,u^2v^2w^2\,du\,dv\,dw = 216abc \iiint_{B_{1,+}} u^2v^2w^2\,du\,dv\,dw.$$

Step 2 — Spherical coordinates on $B_{1,+}$:

Sub **Stage 2: Set $u = \rho\sin\varphi\cos\theta$, $v = \rho\sin\varphi\sin\theta$, $w = \rho\cos\varphi$ with $\rho \in (0,1]$, $\theta \in \left(0,\frac{\pi}{2}\right)$, $\varphi \in \left(0,\frac{\pi}{2}\right)$, and $du\,dv\,dw = \rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$.**

Expanding $u^2v^2w^2 \cdot \rho^2\sin\varphi$:

$$(\rho^2\sin^2\varphi\cos^2\theta)(\rho^2\sin^2\varphi\sin^2\theta)(\rho^2\cos^2\varphi)\cdot\rho^2\sin\varphi = \rho^8\cdot(\sin^2\theta\cos^2\theta)\cdot(\sin^5\varphi\cos^2\varphi).$$

By Fubini's theorem, the triple integral factors into three one-variable integrals:

$$\mathrm{Vol}(V_3) = 216abc\left(\int_0^1 \rho^8\,d\rho\right)\left(\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta\right)\left(\int_0^{\pi/2}\sin^5\varphi\cos^2\varphi\,d\varphi\right).$$

Step 3 — Evaluation of each one-dimensional factor:

(i) **Radial factor:** $\displaystyle\int_0^1 \rho^8\,d\rho = \left[\frac{\rho^9}{9}\right]_0^1 = \frac{1}{9}.$

(ii) **Azimuthal factor:** Using $\sin\theta\cos\theta = \frac{1}{2}\sin(2\theta)$ and $\sin^2(2\theta) = \frac{1-\cos(4\theta)}{2}$:

$$\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta = \frac{1}{8}\int_0^{\pi/2}(1-\cos(4\theta))\,d\theta = \frac{1}{8}\left[\theta - \frac{\sin(4\theta)}{4}\right]_0^{\pi/2} = \frac{\pi}{16}.$$

(iii) **Colatitude factor:** Rewrite $\sin^5\varphi\cos^2\varphi = (1-\cos^2\varphi)^2\cos^2\varphi\sin\varphi$.

Sub **Let $s = \cos\varphi \implies ds = -\sin\varphi\,d\varphi$. Bounds: $\varphi = 0 \implies s = 1$; $\varphi = \frac{\pi}{2} \implies s = 0$.**

$$\int_0^{\pi/2}\sin^5\varphi\cos^2\varphi\,d\varphi = \int_0^1 (1-s^2)^2s^2\,ds = \int_0^1 (s^2 - 2s^4 + s^6)\,ds$$
$$= \left[\frac{s^3}{3} - \frac{2s^5}{5} + \frac{s^7}{7}\right]_0^1 = \frac{1}{3} - \frac{2}{5} + \frac{1}{7} = \frac{35-42+15}{105} = \frac{8}{105}.$$

Multiplying all factors together:

$$\mathrm{Vol}(V_3) = 216abc \cdot \frac{1}{9} \cdot \frac{\pi}{16} \cdot \frac{8}{105} = 24abc \cdot \frac{\pi}{2 \cdot 105} = \frac{12\pi abc}{105} = \frac{4\pi abc}{35}.$$

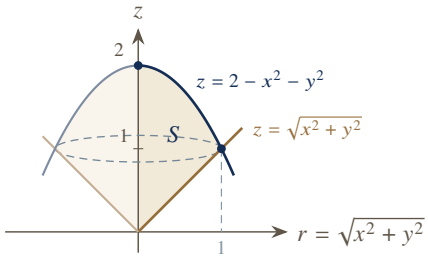
$$\boxed{\mathrm{Vol}(V_3) = \frac{4\pi abc}{35}}$$

> ► **Common Pitfall:**
> Substituting $x = a\rho^3 \sin^3\varphi\cos^3\theta$ directly over $\theta \in [0, 2\pi]$, $\varphi \in [0, \pi]$ without first restricting to the positive octant (where $\sin\theta, \cos\theta, \cos\varphi > 0$) causes the trigonometric integrals of odd powers in $\cos\varphi$ to vanish or change sign across octants because $|J|$ requires $|\sin^5\varphi\cos^2\varphi\sin^2\theta\cos^2\theta|$. Always reduce by eight-octant symmetry first.

> ✓ **Sanity Check:**
> For $t \in [0,1]$, $t^2 \le t \le t^{2/3}$, so in the first octant $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} + \left(\frac{z}{c}\right)^{2/3} \le 1 \implies \frac{|x|}{a} + \frac{|y|}{b} + \frac{|z|}{c} \le 1$. Thus V_3 is strictly inscribed inside the octahedron $\frac{|x|}{a} + \frac{|y|}{b} + \frac{|z|}{c} \le 1$ of volume $8 \cdot \frac{abc}{6} = \frac{4}{3}abc \approx 1.333\,abc$; indeed $\frac{4\pi}{35}abc \approx 0.359\,abc < 1.333\,abc$ (reflecting the inward cusps of the astroidal faces).

Exercise 8

Let $S \subset \mathbb{R}^3$ be the solid bounded by the cone $z = \sqrt{x^2+y^2}$ and the paraboloid $z = 2 - x^2 - y^2$ (see figure below).



1. Compute the volume of S. ★★☆☆☆

Method ***Solid of Revolution Volume in Cylindrical Coordinates***

> ***Why we do this (Way of Thinking):***
> Both bounding surfaces $z = \sqrt{x^2+y^2}$ and $z = 2 - (x^2+y^2)$ depend on (x,y) exclusively through $r = \sqrt{x^2+y^2}$. Equating the two heights determines the intersection circle $r = 1$ (at $z = 1$), so in cylindrical coordinates (r, θ, z) the solid S projects onto the unit disk $0 \le r \le 1, 0 \le \theta \le 2\pi$ with vertical fibers running from the lower cone $z = r$ to the upper paraboloid cap $z = 2 - r^2$.

Execution: Step 1 — Cylindrical domain Δ of the solid S:

Sub **Pass to cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ ($r \ge 0$, $\theta \in [0, 2\pi]$), with volume element $dx\,dy\,dz = r\,dr\,d\theta\,dz$.**

The cone $z = r$ intersects the paraboloid $z = 2 - r^2$ when

$$r = 2 - r^2 \iff r^2 + r - 2 = 0 \iff (r-1)(r+2) = 0.$$

Since $r \ge 0$, the unique radial root is $r = 1$ (giving $z = 1$), and for $0 \le r \le 1$ we have $r \le 2 - r^2$. Thus S corresponds to

$$\Delta = \left\{(r, \theta, z) \in \mathbb{R}^3 : 0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ r \le z \le 2 - r^2\right\}.$$

Step 2 — Iterated triple integral for $\mathrm{Vol}(S)$**:**

$$\begin{aligned}\mathrm{Vol}(S) &= \iiint_S dx\,dy\,dz = \int_0^{2\pi} d\theta \int_0^1 \left(\int_r^{2-r^2} dz\right) r\,dr \\ &= 2\pi \int_0^1 (2 - r^2 - r)\, r\,dr = 2\pi \int_0^1 (2r - r^2 - r^3)\,dr \\ &= 2\pi \left[r^2 - \frac{r^3}{3} - \frac{r^4}{4}\right]_{r=0}^{r=1} = 2\pi \left(1 - \frac{1}{3} - \frac{1}{4}\right) = 2\pi \cdot \frac{12-4-3}{12} = 2\pi \cdot \frac{5}{12} = \frac{5\pi}{6}.\end{aligned}$$

$$\boxed{\mathrm{Vol}(S) = \frac{5\pi}{6}}$$

> ► **Common Pitfall:**
> Omitting the cylindrical Jacobian factor r (integrating $\int_0^1 (2 - r^2 - r)\,dr = \frac{7}{6}$ and multiplying by 2π to get $\frac{7\pi}{3}$) computes 2π times the planar meridian cross-sectional area instead of the three-dimensional volume.

> ✓ **Sanity Check:**
> Decompose S geometrically into two elementary solids meeting at the plane $z = 1$ (radius $R = 1$): the inverted lower cone of height $h_1 = 1$ has volume $\frac{1}{3}\pi R^2 h_1 = \frac{\pi}{3}$, and the upper paraboloidal cap of height $h_2 = 1$ has volume $\frac{1}{2}\pi R^2 h_2 = \frac{\pi}{2}$. Their sum is $\frac{\pi}{3} + \frac{\pi}{2} = \frac{5\pi}{6}$, matching identically.

2. Assume S has density $\rho(x, y, z) = \rho_0\sqrt{x^2 + y^2}$ ($\rho_0 > 0$). Compute the mass M and the center of inertia (x_G, y_G, z_G) of S. ★★★☆☆

Method ***3D Mass, Axisymmetric Parity Reduction, and Vertical Center of Inertia in Cylindrical Coordinates***

> ***Why we do this (Way of Thinking):***
> In cylindrical coordinates, the radial density $\rho(x, y, z) = \rho_0 r$ combines with the Jacobian r into $\rho\,dV = \rho_0 r^2\,dr\,d\theta\,dz$, leaving every required integral polynomial in (r, z). Because both S and $\rho(x, y, z)$ are rotationally invariant around the Oz-axis, the center of inertia G must lie on Oz ($x_G = y_G = 0$), reducing the centroid calculation to two integrals: $M = \iiint_S \rho\,dV$ and $\iiint_S z\rho\,dV$.

Execution: Step 1 — Total mass M**:** Using $\rho(r\cos\theta, r\sin\theta, z) = \rho_0 r$ and $\rho\,dV = \rho_0 r^2\,dr\,d\theta\,dz$ on Δ:

$$\begin{aligned}M &= \iiint_S \rho(x, y, z)\,dx\,dy\,dz = \rho_0 \left(\int_0^{2\pi} d\theta\right) \int_0^1 r^2 \left(\int_r^{2-r^2} dz\right) dr \\ &= 2\pi\rho_0 \int_0^1 r^2(2 - r - r^2)\,dr = 2\pi\rho_0 \int_0^1 (2r^2 - r^3 - r^4)\,dr \\ &= 2\pi\rho_0 \left[\frac{2r^3}{3} - \frac{r^4}{4} - \frac{r^5}{5}\right]_{r=0}^{r=1} = 2\pi\rho_0 \left(\frac{2}{3} - \frac{1}{4} - \frac{1}{5}\right) \\ &= 2\pi\rho_0 \cdot \frac{40 - 15 - 12}{60} = 2\pi\rho_0 \cdot \frac{13}{60} = \frac{13\pi\rho_0}{30}.\end{aligned}$$

Step 2 — Horizontal coordinates x_G **and** y_G **by reflection symmetry:** The solid S is invariant under $(x, y, z) \mapsto (-x, y, z)$ and $(x, y, z) \mapsto (x, -y, z)$, while $\rho(x, y, z) = \rho_0\sqrt{x^2 + y^2}$ is even in x and even in y. Consequently, $x\,\rho(x, y, z)$ is odd in x and $y\,\rho(x, y, z)$ is odd in y (equivalently, $\int_0^{2\pi} \cos\theta\,d\theta = \int_0^{2\pi} \sin\theta\,d\theta = 0$), so

$$\iiint_S x\,\rho\,dV = 0, \qquad \iiint_S y\,\rho\,dV = 0 \quad \Longrightarrow \quad x_G = 0, \quad y_G = 0.$$

Step 3 — Vertical coordinate z_G: Computing the first static moment $\iiint_S z\,\rho\,dV$:

$$\begin{aligned}\iiint_S z\,\rho(x,y,z)\,dx\,dy\,dz &= \rho_0\left(\int_0^{2\pi} d\theta\right)\int_0^1 r^2\left(\int_r^{2-r^2} z\,dz\right)dr\\ &= 2\pi\rho_0\int_0^1 r^2\left[\frac{z^2}{2}\right]_{z=r}^{z=2-r^2}dr = \pi\rho_0\int_0^1 r^2\Big((2-r^2)^2-r^2\Big)\,dr\\ &= \pi\rho_0\int_0^1 r^2(4-4r^2+r^4-r^2)\,dr = \pi\rho_0\int_0^1 (4r^2-5r^4+r^6)\,dr\\ &= \pi\rho_0\left[\frac{4r^3}{3}-r^5+\frac{r^7}{7}\right]_{r=0}^{r=1} = \pi\rho_0\left(\frac{4}{3}-1+\frac{1}{7}\right)\\ &= \pi\rho_0\cdot\frac{28-21+3}{21} = \frac{10\pi\rho_0}{21}.\end{aligned}$$

Dividing by $M = \dfrac{13\pi\rho_0}{30}$:

$$z_G = \frac{1}{M}\iiint_S z\,\rho\,dV = \frac{10\pi\rho_0/21}{13\pi\rho_0/30} = \frac{10}{21}\cdot\frac{30}{13} = \frac{10\cdot 10}{7\cdot 13} = \frac{100}{91}.$$

$$\boxed{M = \frac{13\pi\rho_0}{30}, \qquad (x_G, y_G, z_G) = \left(0,\ 0,\ \frac{100}{91}\right)}$$

► **Common Pitfall:**
Notice that $\rho(r\cos\theta, r\sin\theta, z) = \rho_0 r$ contributes one factor of r and the cylindrical Jacobian $dx\,dy\,dz = r\,dr\,d\theta\,dz$ contributes a *second* factor of r, making $\rho\,dV = \rho_0 r^2\,dr\,d\theta\,dz$. Using only r^1 computes the homogeneous centroid $z_{G,\text{hom}} = \frac{7/6}{5/6} = \frac{7}{5}$ instead of $\frac{100}{91}$.

✓ **Sanity Check:**
Check dimensions and geometric location: $z_G = \frac{100}{91} \approx 1.0989$ is independent of ρ_0 and lies strictly inside the vertical span $[0,2]$ of S, very close to the widest rim $z = 1$ ($r = 1$) because the density $\rho = \rho_0 r$ vanishes along the axis $r = 0$ and concentrates mass near the outer rim $r = 1$ (pulling z_G down from the homogeneous value 1.4 toward 1).

3. For the same density, compute the moments of inertia I_{xOy}, I_{yOz}, I_{xOz} and I_{Ox}, I_{Oy}, I_{Oz} of S. ★★★☆☆

Method *3D Planar-to-Axial Moment of Inertia Relations and Axisymmetric Equipartition*

Why we do this (Way of Thinking):
Rather than computing six separate triple integrals, use the three structural identities connecting planar moments ($I_{yOz} = \iiint_S x^2\rho\,dV$, $I_{xOz} = \iiint_S y^2\rho\,dV$, $I_{xOy} = \iiint_S z^2\rho\,dV$) to axial moments:

$$I_{Oz} = I_{yOz} + I_{xOz}, \qquad I_{Ox} = I_{xOz} + I_{xOy}, \qquad I_{Oy} = I_{yOz} + I_{xOy}.$$

Because S and $\rho = \rho_0 r$ are invariant under $(x,y) \leftrightarrow (y,x)$, we have $I_{yOz} = I_{xOz} = \frac{1}{2}I_{Oz}$, where $I_{Oz} = \iiint_S r^2\rho\,dV$ has no θ-dependence at all. Thus *all six* moments of inertia follow from just two radial integrals: I_{Oz} and I_{xOy}.

Execution: Step 1 — Axial moment I_{Oz} and planar moments I_{yOz}, I_{xOz}: Since $x^2 + y^2 = r^2$ and

$\rho\,dV = \rho_0 r^2\,dr\,d\theta\,dz$:

$$\begin{aligned} I_{Oz} &= \iiint_S (x^2+y^2)\,\rho(x,y,z)\,dx\,dy\,dz = \rho_0 \left(\int_0^{2\pi} d\theta\right)\int_0^1 r^4 \left(\int_r^{2-r^2} dz\right) dr \\ &= 2\pi\rho_0 \int_0^1 r^4\big(2-r-r^2\big)\,dr = 2\pi\rho_0 \int_0^1 \big(2r^4 - r^5 - r^6\big)\,dr \\ &= 2\pi\rho_0 \left[\frac{2r^5}{5} - \frac{r^6}{6} - \frac{r^7}{7}\right]_{r=0}^{r=1} = 2\pi\rho_0 \left(\frac{2}{5} - \frac{1}{6} - \frac{1}{7}\right) \\ &= 2\pi\rho_0 \cdot \frac{84-35-30}{210} = 2\pi\rho_0 \cdot \frac{19}{210} = \frac{19\pi\rho_0}{105}. \end{aligned}$$

By symmetry under $(x,y,z) \mapsto (y,x,z)$, $I_{yOz} = \iiint_S x^2\rho\,dV = \iiint_S y^2\rho\,dV = I_{xOz}$. Since $I_{yOz} + I_{xOz} = \iiint_S (x^2+y^2)\rho\,dV = I_{Oz}$, we obtain

$$I_{yOz} = I_{xOz} = \frac{1}{2} I_{Oz} = \frac{19\pi\rho_0}{210}.$$

Step 2 — Planar moment I_{xOy}: Evaluating $I_{xOy} = \iiint_S z^2\rho\,dV$:

$$\begin{aligned} I_{xOy} &= \rho_0 \left(\int_0^{2\pi} d\theta\right)\int_0^1 r^2 \left(\int_r^{2-r^2} z^2\,dz\right) dr = 2\pi\rho_0 \int_0^1 r^2 \left[\frac{z^3}{3}\right]_{z=r}^{z=2-r^2} dr \\ &= \frac{2\pi\rho_0}{3} \int_0^1 r^2\Big((2-r^2)^3 - r^3\Big)\,dr. \end{aligned}$$

Expanding $(2-r^2)^3 - r^3 = 8 - 12r^2 + 6r^4 - r^6 - r^3$ and multiplying by r^2:

$$\begin{aligned} I_{xOy} &= \frac{2\pi\rho_0}{3} \int_0^1 \big(8r^2 - 12r^4 - r^5 + 6r^6 - r^8\big)\,dr \\ &= \frac{2\pi\rho_0}{3} \left[\frac{8r^3}{3} - \frac{12r^5}{5} - \frac{r^6}{6} + \frac{6r^7}{7} - \frac{r^9}{9}\right]_{r=0}^{r=1} \\ &= \frac{2\pi\rho_0}{3} \left(\frac{8}{3} - \frac{12}{5} - \frac{1}{6} + \frac{6}{7} - \frac{1}{9}\right). \end{aligned}$$

Expressing all five fractions over $\operatorname{lcm}(3,5,6,7,9) = 630$:

$$\frac{8}{3} - \frac{12}{5} - \frac{1}{6} + \frac{6}{7} - \frac{1}{9} = \frac{1680 - 1512 - 105 + 540 - 70}{630} = \frac{533}{630},$$

and therefore

$$I_{xOy} = \frac{2\pi\rho_0}{3} \cdot \frac{533}{630} = \frac{533\pi\rho_0}{945}.$$

Step 3 — Axial moments I_{Ox} and I_{Oy}: Using $I_{Ox} = \iiint_S (y^2+z^2)\rho\,dV = I_{xOz} + I_{xOy}$ and $I_{Oy} = \iiint_S (x^2+z^2)\rho\,dV = I_{yOz} + I_{xOy}$ (with common denominator $\operatorname{lcm}(210,945) = 1890 = 210\times 9 = 945\times 2$):

$$I_{Ox} = I_{Oy} = \frac{19\pi\rho_0}{210} + \frac{533\pi\rho_0}{945} = \frac{19\times 9 + 533\times 2}{1890}\,\pi\rho_0 = \frac{171 + 1066}{1890}\,\pi\rho_0 = \frac{1237\pi\rho_0}{1890}.$$

$$\boxed{\begin{aligned} &I_{xOy} = \frac{533\pi\rho_0}{945}, \qquad I_{yOz} = I_{xOz} = \frac{19\pi\rho_0}{210}, \\ &I_{Ox} = I_{Oy} = \frac{1237\pi\rho_0}{1890}, \quad I_{Oz} = \frac{19\pi\rho_0}{105} \end{aligned}}$$

► **Common Pitfall:**
Do not confuse 3D planar vs. axial moment formulas with 2D formulas: in 3D, the distance squared from (x, y, z) to the *Ox-axis* is $y^2 + z^2$ (not y^2), whereas y^2 is the distance squared to the *xOz-plane* (I_{xOz}). Thus $I_{Ox} = I_{xOz} + I_{xOy}$, not I_{xOz} alone.

✓ **Sanity Check:**
Verify the universal 3D trace identity $I_{Ox} + I_{Oy} + I_{Oz} = 2\big(I_{xOy} + I_{yOz} + I_{xOz}\big)$: the left-hand side is $\left(\frac{1237}{1890} + \frac{1237}{1890} + \frac{342}{1890}\right)\pi\rho_0 = \frac{2816}{1890}\pi\rho_0$, and the right-hand side is $2\left(\frac{1066}{1890} + \frac{171}{1890} + \frac{171}{1890}\right)\pi\rho_0 = 2 \cdot \frac{1408}{1890}\pi\rho_0 = \frac{2816}{1890}\pi\rho_0$, matching identically. Also, since $r \leq 1$ on S, $I_{Oz} = \iiint_S r^2\rho\, dV < M = \frac{13\pi\rho_0}{30} \approx 0.433\,\pi\rho_0$; indeed $\frac{19}{105} \approx 0.181 < 0.433$.