Analysis 3
Exercise Set 2

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Vector Calculus — Detailed Correction

Analysis 3 — Detailed Correction – Exercise Set 2

Focus on the Way of Thinking, Explicit Substitutions, and Step-by-Step Course Execution

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How to Use This Correction

Each solution follows a strict four-stage analytical structure designed to train exam intuition: **Method** (identifies the exact course technique) ⟶ **Why we do this** (explains the mental train of thought and algebraic clues) ⟶ **Execution** (complete line-by-line derivation with zero skipped steps and explicit **Sub** callouts) ⟶ **Pitfall / Sanity Check** (where students lose points and how to verify your answer).

Difficulty: ★☆☆☆☆ Direct ★★☆☆☆ Standard ★★★☆☆ Exam level ★★★★☆ Tricky ★★★★★ Synthesis

Exercise 1

Throughout this exercise, $\mathbb{R}^3$ is equipped with its canonical orthonormal basis $(\vec{i},\vec{j},\vec{k})$, and we set $\vec{r}=x\,\vec{i}+y\,\vec{j}+z\,\vec{k}$ and $r=\|\vec{r}\|=\sqrt{x^2+y^2+z^2}$ on $U=\mathbb{R}^3\setminus\{(0,0,0)\}$.

1. Let $\varphi(x,y,z)=x^2y-yz^3$ and $\vec{A}(x,y,z)=yz\,\vec{i}-xz\,\vec{j}+x^2y\,\vec{k}$ on $\mathbb{R}^3$. Compute $\nabla\varphi$, $\Delta\varphi$, $\operatorname{div}\vec{A}$, $\operatorname{rot}\vec{A}$, $\operatorname{div}(\operatorname{rot}\vec{A})$, and $\operatorname{rot}(\nabla\varphi)$. ★☆☆☆☆

Method *Direct Cartesian evaluation of first- and second-order differential operators*

> ***Why we do this (Way of Thinking):***
> In Cartesian coordinates, $\nabla\varphi$ collects the three first-order partial derivatives of φ, $\Delta\varphi=\operatorname{div}(\nabla\varphi)$ sums the three unmixed second-order partial derivatives, $\operatorname{div}\vec{A}$ sums the diagonal partial derivatives $\partial_xA_x+\partial_yA_y+\partial_zA_z$, and $\operatorname{rot}\vec{A}=\nabla\wedge\vec{A}$ evaluates the antisymmetric off-diagonal partial derivatives. Computing $\operatorname{div}(\operatorname{rot}\vec{A})$ and $\operatorname{rot}(\nabla\varphi)$ directly from those intermediate expressions also provides an independent algebraic check via Schwarz's mixed-derivative symmetry on $C^2(\mathbb{R}^3)$.

Execution:

- **Step 1 (Gradient $\nabla\varphi$ and Laplacian $\Delta\varphi$):** Differentiating $\varphi(x,y,z)=x^2y-yz^3$ with respect to each variable while holding the remaining two variables fixed gives:

$$\frac{\partial\varphi}{\partial x}=2xy,\qquad \frac{\partial\varphi}{\partial y}=x^2-z^3,\qquad \frac{\partial\varphi}{\partial z}=-3yz^2.$$

Therefore, the gradient vector field is

$$\nabla\varphi(x,y,z)=2xy\,\vec{i}+(x^2-z^3)\,\vec{j}-3yz^2\,\vec{k}.$$

Differentiating each component a second time with respect to the same variable gives:

$$\frac{\partial^2\varphi}{\partial x^2}=\frac{\partial}{\partial x}(2xy)=2y,\qquad \frac{\partial^2\varphi}{\partial y^2}=\frac{\partial}{\partial y}(x^2-z^3)=0,\qquad \frac{\partial^2\varphi}{\partial z^2}=\frac{\partial}{\partial z}(-3yz^2)=-6yz.$$

Summing the three second-order partial derivatives yields the Laplacian:

$$\Delta\varphi(x,y,z)=\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}=2y+0-6yz=2y(1-3z).$$

- **Step 2 (Divergence $\operatorname{div}\vec{A}$ and curl $\operatorname{rot}\vec{A}$):** Write $\vec{A}=A_x\,\vec{i}+A_y\,\vec{j}+A_z\,\vec{k}$ with $A_x(x,y,z)=yz$, $A_y(x,y,z)=-xz$, and $A_z(x,y,z)=x^2y$. For the divergence, each component is independent of its corresponding coordinate:

$$\frac{\partial A_x}{\partial x}=\frac{\partial}{\partial x}(yz)=0,\qquad \frac{\partial A_y}{\partial y}=\frac{\partial}{\partial y}(-xz)=0,\qquad \frac{\partial A_z}{\partial z}=\frac{\partial}{\partial z}(x^2y)=0,$$

whence $\operatorname{div}\vec{A}(x,y,z) = 0+0+0 = 0$. For the curl

$$\operatorname{rot}\vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\vec{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\vec{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\vec{k},$$

we compute all six off-diagonal first-order partial derivatives explicitly:

$$\begin{aligned} \frac{\partial A_z}{\partial y} &= \frac{\partial}{\partial y}(x^2 y) = x^2, & \frac{\partial A_y}{\partial z} &= \frac{\partial}{\partial z}(-xz) = -x, \\ \frac{\partial A_x}{\partial z} &= \frac{\partial}{\partial z}(yz) = y, & \frac{\partial A_z}{\partial x} &= \frac{\partial}{\partial x}(x^2 y) = 2xy, \\ \frac{\partial A_y}{\partial x} &= \frac{\partial}{\partial x}(-xz) = -z, & \frac{\partial A_x}{\partial y} &= \frac{\partial}{\partial y}(yz) = z. \end{aligned}$$

Substituting these six derivatives into the curl expression gives:

$$\operatorname{rot}\vec{A}(x,y,z) = \big(x^2-(-x)\big)\vec{i} + (y-2xy)\vec{j} + (-z-z)\vec{k} = (x^2+x)\,\vec{i} + (y-2xy)\,\vec{j} - 2z\,\vec{k}.$$

- **Step 3 (Second-order compositions $\operatorname{div}(\operatorname{rot}\vec{A})$ and $\operatorname{rot}(\nabla\varphi)$):** Taking the divergence of $\operatorname{rot}\vec{A} = (x^2+x)\,\vec{i} + (y-2xy)\,\vec{j} - 2z\,\vec{k}$ directly gives:

$$\operatorname{div}(\operatorname{rot}\vec{A}) = \frac{\partial}{\partial x}(x^2+x) + \frac{\partial}{\partial y}(y-2xy) + \frac{\partial}{\partial z}(-2z) = (2x+1) + (1-2x) - 2 = 0.$$

Next, writing $\nabla\varphi = G_x\,\vec{i} + G_y\,\vec{j} + G_z\,\vec{k}$ with $G_x = 2xy$, $G_y = x^2 - z^3$, and $G_z = -3yz^2$, we evaluate $\operatorname{rot}(\nabla\varphi)$ component by component:

$$\begin{aligned} (\operatorname{rot}(\nabla\varphi))_x &= \frac{\partial G_z}{\partial y} - \frac{\partial G_y}{\partial z} = \frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(x^2 - z^3) = -3z^2 - (-3z^2) = 0, \\ (\operatorname{rot}(\nabla\varphi))_y &= \frac{\partial G_x}{\partial z} - \frac{\partial G_z}{\partial x} = \frac{\partial}{\partial z}(2xy) - \frac{\partial}{\partial x}(-3yz^2) = 0 - 0 = 0, \\ (\operatorname{rot}(\nabla\varphi))_z &= \frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y} = \frac{\partial}{\partial x}(x^2 - z^3) - \frac{\partial}{\partial y}(2xy) = 2x - 2x = 0, \end{aligned}$$

so $\operatorname{rot}(\nabla\varphi)(x,y,z) = \vec{0}$. Both direct evaluations agree with the general identities $\operatorname{div}(\operatorname{rot}\vec{A}) = 0$ and $\operatorname{rot}(\nabla\varphi) = \vec{0}$ valid for C^2 fields by Schwarz's theorem.

$\nabla\varphi = 2xy\,\vec{i} + (x^2 - z^3)\,\vec{j} - 3yz^2\,\vec{k}$,	$\Delta\varphi = 2y(1-3z)$,
$\operatorname{div}\vec{A} = 0$,	$\operatorname{rot}\vec{A} = (x^2+x)\,\vec{i} + (y-2xy)\,\vec{j} - 2z\,\vec{k}$,
$\operatorname{div}(\operatorname{rot}\vec{A}) = 0$,	$\operatorname{rot}(\nabla\varphi) = \vec{0}$

> ✓ **Sanity Check:**
> Checking $\operatorname{div}(\operatorname{rot}\vec{A}) = (2x+1) + (1-2x) - 2 = 0$ immediately catches any cofactor sign error on the middle $\vec{j}$-component of $\operatorname{rot}\vec{A}$: had $\partial_x A_z - \partial_z A_x = 2xy - y$ been used instead of $\partial_z A_x - \partial_x A_z = y - 2xy$, the divergence sum would have been $(2x+1) + (2x-1) - 2 = 4x - 2 \neq 0$.

2. Let $g \in C^2(]0,+\infty[,\mathbb{R})$ and define $u(x,y,z) = g(r)$ on U.

(a) Prove that $\nabla u(x,y,z) = \dfrac{g'(r)}{r}\,\vec{r}$ and $\Delta u(x,y,z) = g''(r) + \dfrac{2}{r}\,g'(r)$ on U. ★★☆☆☆

Method *Multivariable chain rule and product differentiation on the Euclidean norm* $r =$

$\sqrt{x^2+y^2+z^2}$

> ***Why we do this (Way of Thinking):***
> On $U=\mathbb{R}^3\setminus\{(0,0,0)\}$, the polynomial $s(x,y,z)=x^2+y^2+z^2$ satisfies $s>0$, so $r=\sqrt{s}$ is of class $C^\infty(U)$ with values in $]0,+\infty[$ and the composite $u=g\circ r$ belongs to $C^2(U)$. Expressing $\partial_x u$ as the product $x\cdot\frac{g'(r)}{r}$ isolates the radial factor $\frac{g'(r)}{r}$, so that summing $\partial_{xx}u+\partial_{yy}u+\partial_{zz}u$ reconstructs $x^2+y^2+z^2=r^2$ and simplifies the denominator r^3.

Execution:

- **Step 1 (First partial derivatives of r and ∇u):** Differentiating $r(x,y,z)=(x^2+y^2+z^2)^{1/2}$ on U gives:

$$\frac{\partial r}{\partial x}=\frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2x)=\frac{x}{r},$$
$$\frac{\partial r}{\partial y}=\frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2y)=\frac{y}{r},$$
$$\frac{\partial r}{\partial z}=\frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2z)=\frac{z}{r}.$$

  Applying the single-variable chain rule to $u(x,y,z)=g(r(x,y,z))$ along each coordinate axis yields:

$$\frac{\partial u}{\partial x}=g'(r)\,\frac{\partial r}{\partial x}=\frac{g'(r)}{r}\,x,\qquad \frac{\partial u}{\partial y}=g'(r)\,\frac{\partial r}{\partial y}=\frac{g'(r)}{r}\,y,\qquad \frac{\partial u}{\partial z}=g'(r)\,\frac{\partial r}{\partial z}=\frac{g'(r)}{r}\,z.$$

  Collecting all three components into ∇u produces

$$\nabla u(x,y,z)=\frac{g'(r)}{r}\,(x\,\vec{i}+y\,\vec{j}+z\,\vec{k})=\frac{g'(r)}{r}\,\vec{r}.$$

- **Step 2 (Second partial derivatives and Δu):** By the quotient rule on $]0,+\infty[$, the function $q(r)=\frac{g'(r)}{r}$ is of class $C^1(]0,+\infty[)$ with derivative

$$q'(r)=\frac{d}{dr}\left(\frac{g'(r)}{r}\right)=\frac{rg''(r)-g'(r)}{r^2}.$$

  Differentiating $\frac{\partial u}{\partial x}=x\,q(r)$ with respect to x by the product and chain rules gives:

$$\frac{\partial^2 u}{\partial x^2}=1\cdot q(r)+x\cdot q'(r)\,\frac{\partial r}{\partial x}=\frac{g'(r)}{r}+x\cdot\frac{rg''(r)-g'(r)}{r^2}\cdot\frac{x}{r}=\frac{g'(r)}{r}+\frac{rg''(r)-g'(r)}{r^3}\,x^2.$$

  Differentiating $\frac{\partial u}{\partial y}=y\,q(r)$ with respect to y and $\frac{\partial u}{\partial z}=z\,q(r)$ with respect to z by the exact same calculation gives:

$$\frac{\partial^2 u}{\partial y^2}=\frac{g'(r)}{r}+\frac{rg''(r)-g'(r)}{r^3}\,y^2,\qquad \frac{\partial^2 u}{\partial z^2}=\frac{g'(r)}{r}+\frac{rg''(r)-g'(r)}{r^3}\,z^2.$$

  Adding the three expressions and using $x^2+y^2+z^2=r^2$ on U:

$$\begin{aligned}\Delta u(x,y,z)&=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\\&=\frac{3g'(r)}{r}+\frac{rg''(r)-g'(r)}{r^3}\,(x^2+y^2+z^2)\\&=\frac{3g'(r)}{r}+\frac{rg''(r)-g'(r)}{r^3}\,r^2=\frac{3g'(r)}{r}+g''(r)-\frac{g'(r)}{r}=g''(r)+\frac{2}{r}\,g'(r).\end{aligned}$$

$$\boxed{\nabla u(x,y,z)=\frac{g'(r)}{r}\,\vec{r},\qquad \Delta u(x,y,z)=g''(r)+\frac{2}{r}\,g'(r)\quad\text{on } U}$$

(b) Determine all functions $g \in C^2(]0,+\infty[,\mathbb{R})$ such that $\Delta u = 0$ on U. ★★★☆☆

Method *Reduction of the radial Euler-type ODE via the integrating factor $r^2 > 0$*

> ***Why we do this (Way of Thinking):***
> As (x,y,z) varies over U, the radius $r = \|\vec{r}\|$ surjects onto $]0,+\infty[$ (for instance $(r,0,0) \in U$ for every $r > 0$). Thus $\Delta u = 0$ on U is equivalent to the second-order linear ODE $g''(r) + \frac{2}{r}g'(r) = 0$ on $]0,+\infty[$. Multiplying by $r^2 > 0$ turns $r^2 g''(r) + 2rg'(r)$ into the exact derivative $\frac{d}{dr}\big(r^2 g'(r)\big)$, allowing two direct integrations on the connected interval $]0,+\infty[$.

Execution: By Question 2(a) and surjectivity of $(x,y,z) \mapsto r$ from U onto $]0,+\infty[$:

$$\Delta u(x,y,z) = 0 \quad \forall (x,y,z) \in U \iff g''(r) + \frac{2}{r}\,g'(r) = 0 \quad \forall r \in]0,+\infty[.$$

Since $r^2 > 0$ for every $r \in]0,+\infty[$, multiplying the equation by r^2 preserves equivalence:

$$r^2 g''(r) + 2r\,g'(r) = 0 \iff \frac{d}{dr}\big(r^2 g'(r)\big) = 0 \qquad \forall r \in]0,+\infty[.$$

Sub **First integration on the connected interval $]0,+\infty[$: $r^2 g'(r) = C_1$ with $C_1 \in \mathbb{R}$**

Dividing both sides by $r^2 > 0$ yields

$$g'(r) = \frac{C_1}{r^2} = C_1 r^{-2} \qquad \forall r \in]0,+\infty[.$$

Sub **Second integration on $]0,+\infty[$: $\int r^{-2}\,dr = -r^{-1} + C_2$ with $C_2 \in \mathbb{R}$**

Integrating on $]0,+\infty[$ gives

$$g(r) = -\frac{C_1}{r} + C_2.$$

Setting $A = -C_1 \in \mathbb{R}$ and $B = C_2 \in \mathbb{R}$, every solution has the form $g(r) = \frac{A}{r} + B$, and conversely any such function belongs to $C^\infty(]0,+\infty[,\mathbb{R}) \subset C^2(]0,+\infty[,\mathbb{R})$ and satisfies $\frac{d}{dr}\big(r^2(-Ar^{-2})\big) = \frac{d}{dr}(-A) = 0$.

$$\boxed{g(r) = \frac{A}{r} + B, \qquad (A,B) \in \mathbb{R}^2}$$

> ▸ **Common Pitfall:**
> Do not confuse three-dimensional radial harmonic functions with their two-dimensional counterparts: in $\mathbb{R}^3 \setminus \{(0,0,0)\}$, summing three coordinate derivatives contributes $\frac{3g'(r)}{r} - \frac{g'(r)}{r} = \frac{2}{r}g'(r)$ and gives $g(r) = \frac{A}{r} + B$, whereas in $\mathbb{R}^2 \setminus \{(0,0)\}$ summing two derivatives gives $g''(r) + \frac{1}{r}g'(r) = 0$ and logarithmic solutions $A \ln r + B$. Moreover, when $A \neq 0$, $u(x,y,z) = \frac{A}{r} + B$ is harmonic only on the punctured space U, since $r = 0$ is singular.

3. For $\alpha \in \mathbb{R}$, define $\vec{F}_\alpha(x,y,z) = r^\alpha\,\vec{r}$ on U.

(a) Compute $\operatorname{div}\vec{F}_\alpha$ and $\operatorname{rot}\vec{F}_\alpha$ on U, and determine the value of α for which $\operatorname{div}\vec{F}_\alpha = 0$ on U. ★★☆☆☆

Method *Differential operator product identities applied to $\vec{F}_\alpha = r^\alpha\,\vec{r}$*

> ***Why we do this (Way of Thinking):***
> Factoring $\vec{F}_\alpha = \psi_\alpha\,\vec{r}$ with the radial scalar multiplier $\psi_\alpha(x,y,z) = r^\alpha \in C^\infty(U)$ lets us apply the product identities $\operatorname{div}(\psi_\alpha\vec{r}) = \psi_\alpha \operatorname{div}\vec{r} + \nabla\psi_\alpha \cdot \vec{r}$ and $\operatorname{rot}(\psi_\alpha\vec{r}) = \psi_\alpha \operatorname{rot}\vec{r} + \nabla\psi_\alpha \wedge \vec{r}$, where $\nabla(r^\alpha)$ follows immediately from Question 2(a) with $g(r) = r^\alpha$.

Execution:

- **Step 1 (Operators on $\vec{r}$ and $\psi_\alpha = r^\alpha$):** For the position vector field $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$:

$$\operatorname{div}\vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3, \qquad \operatorname{rot}\vec{r} = (0-0)\vec{i} + (0-0)\vec{j} + (0-0)\vec{k} = \vec{0}.$$

Applying Question 2(a) to $\psi_\alpha(x,y,z) = r^\alpha$ on U (so $g(r) = r^\alpha$ and $g'(r) = \alpha r^{\alpha-1}$) gives

$$\nabla(r^\alpha) = \frac{\alpha r^{\alpha-1}}{r}\,\vec{r} = \alpha r^{\alpha-2}\,\vec{r}.$$

- **Step 2 (Divergence and curl of $\vec{F}_\alpha$):** Using $\operatorname{div}(\psi_\alpha\vec{r}) = \psi_\alpha \operatorname{div}\vec{r} + \nabla\psi_\alpha\cdot\vec{r}$ together with $\vec{r}\cdot\vec{r} = \|\vec{r}\|^2 = r^2$:

$$\operatorname{div}\vec{F}_\alpha(x,y,z) = r^\alpha(3) + \left(\alpha r^{\alpha-2}\,\vec{r}\right)\cdot\vec{r} = 3r^\alpha + \alpha r^{\alpha-2}r^2 = (\alpha+3)\,r^\alpha.$$

(Equivalently, in Cartesian components, $\partial_x(xr^\alpha) = r^\alpha + x\cdot\alpha r^{\alpha-1}\frac{x}{r} = r^\alpha + \alpha x^2 r^{\alpha-2}$, and summing with $\partial_y(yr^\alpha) = r^\alpha + \alpha y^2 r^{\alpha-2}$ and $\partial_z(zr^\alpha) = r^\alpha + \alpha z^2 r^{\alpha-2}$ gives $3r^\alpha + \alpha(x^2+y^2+z^2)r^{\alpha-2} = (\alpha+3)r^\alpha$.) Next, using $\operatorname{rot}(\psi_\alpha\vec{r}) = \psi_\alpha \operatorname{rot}\vec{r} + \nabla\psi_\alpha \wedge \vec{r}$ and $\vec{r}\wedge\vec{r} = \vec{0}$:

$$\operatorname{rot}\vec{F}_\alpha(x,y,z) = r^\alpha\,\vec{0} + \left(\alpha r^{\alpha-2}\,\vec{r}\right)\wedge\vec{r} = \alpha r^{\alpha-2}\,(\vec{r}\wedge\vec{r}) = \vec{0}.$$

- **Step 3 (Divergence-free exponent):** At every point $(x,y,z) \in U$, we have $r > 0$ and hence $r^\alpha > 0$. Consequently:

$$\operatorname{div}\vec{F}_\alpha = 0 \text{ on } U \iff (\alpha+3)\,r^\alpha = 0 \text{ on } U \iff \alpha + 3 = 0 \iff \alpha = -3,$$

which singles out the inverse-square Newtonian and Coulombian radial field $\vec{F}_{-3}(x,y,z) = \frac{\vec{r}}{r^3}$.

$$\begin{aligned} &\operatorname{div}\vec{F}_\alpha = (\alpha+3)\,r^\alpha, && \operatorname{rot}\vec{F}_\alpha = \vec{0} \quad \text{on } U,\\ &\operatorname{div}\vec{F}_\alpha = 0 \text{ on } U && \iff \alpha = -3 \end{aligned}$$

(b) Show that $\vec{F}_\alpha$ is conservative on U for every $\alpha \in \mathbb{R}$, and determine all its scalar potentials on U (distinguishing the case $\alpha = -2$). ★★★☆☆

Method ***Explicit radial antiderivative $V_\alpha(x,y,z) = h(r)$ and connectedness of $U = \mathbb{R}^3 \setminus \{(0,0,0)\}$***

> ***Why we do this (Way of Thinking):***
> To prove that $\vec{F}_\alpha$ is conservative on U without invoking any topological hypothesis on the punctured domain U, it suffices to exhibit an explicit function $V_\alpha \in C^1(U,\mathbb{R})$ satisfying $\nabla V_\alpha = \vec{F}_\alpha$ on U. Because $\vec{F}_\alpha$ is purely radial, testing the ansatz $V_\alpha(x,y,z) = h(r)$ via Question 2(a) reduces $\nabla V_\alpha = r^\alpha\,\vec{r}$ to the single-variable equation $h'(r) = r^{\alpha+1}$, where the critical exponent $\alpha + 1 = -1$ ($\alpha = -2$) separates power potentials from the logarithmic potential.

Execution:

- **Step 1 (Construction of a particular radial potential V_α):** Let $h \in C^1(]0,+\infty[,\mathbb{R})$ and set $V_\alpha(x,y,z) = h(r)$ on U. By Question 2(a), $\nabla V_\alpha(x,y,z) = \frac{h'(r)}{r}\,\vec{r}$ on U. Since $\vec{r} \neq \vec{0}$ everywhere on U, we have:

$$\nabla V_\alpha(x,y,z) = \vec{F}_\alpha(x,y,z) \quad \forall (x,y,z)\in U \iff \frac{h'(r)}{r} = r^\alpha \quad \forall r > 0 \iff h'(r) = r^{\alpha+1} \quad \forall r > 0.$$

We distinguish two cases according to whether $\alpha + 1 = -1$:

– **Case 1 ($\alpha \neq -2$, so $\alpha + 2 \neq 0$):**

Sub **Power antiderivative on $]0,+\infty[$: $h(r) = \int r^{\alpha+1}\,dr = \dfrac{r^{\alpha+2}}{\alpha+2}$**

Defining $V_\alpha(x,y,z) = \dfrac{r^{\alpha+2}}{\alpha+2} = \dfrac{(x^2+y^2+z^2)^{(\alpha+2)/2}}{\alpha+2}$ yields $V_\alpha \in C^\infty(U,\mathbb{R})$ with $\nabla V_\alpha = r^\alpha\,\vec{r} = \vec{F}_\alpha$ on U.

– **Case 2 ($\alpha = -2$, so $\alpha + 1 = -1$):**

Sub **Logarithmic antiderivative on $]0, +\infty[$: $h(r) = \int \frac{dr}{r} = \ln r$**

Defining $V_{-2}(x, y, z) = \ln r = \frac{1}{2}\ln(x^2 + y^2 + z^2)$ yields $V_{-2} \in C^\infty(U, \mathbb{R})$ with $\nabla V_{-2} = \frac{1/r}{r}\vec{r} = r^{-2}\vec{r} = \vec{F}_{-2}$ on U.

In both cases, $\vec{F}_\alpha = \nabla V_\alpha$ on U with $V_\alpha \in C^1(U, \mathbb{R})$, which proves by definition that $\vec{F}_\alpha$ is conservative on U for every $\alpha \in \mathbb{R}$.

- **Step 2 (Determination of all scalar potentials on U):** In $\mathbb{R}^3$ (dimension $3 \geq 2$), the punctured space $U = \mathbb{R}^3 \setminus \{(0,0,0)\}$ is open and path-connected: any two points $P, Q \in U$ are joined in U either by the straight segment $[P, Q]$ when $(0,0,0) \notin [P, Q]$, or by the broken line $[P, R] \cup [R, Q]$ through any point $R \notin \mathbb{R}P$ when $(0,0,0) \in [P, Q]$. Consequently, a function $V \in C^1(U, \mathbb{R})$ satisfies $\nabla V = \vec{F}_\alpha$ on U if and only if

$$\nabla(V - V_\alpha) = \vec{0} \quad \text{on } U \iff V(x,y,z) - V_\alpha(x,y,z) = C \quad \forall (x,y,z) \in U$$

for some constant $C \in \mathbb{R}$.

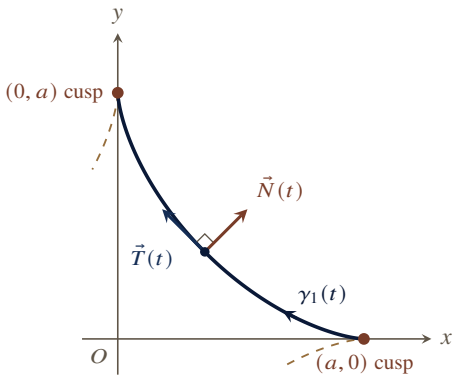
$$V(x,y,z) = \begin{cases} \dfrac{r^{\alpha+2}}{\alpha+2} + C, & \text{if } \alpha \neq -2, \\ \ln r + C = \dfrac{1}{2}\ln(x^2+y^2+z^2) + C, & \text{if } \alpha = -2, \end{cases} \qquad C \in \mathbb{R}$$

✓ **Sanity Check:**
Evaluating Case 1 at $\alpha = -3$ gives $V(x,y,z) = -\frac{1}{r} + C$. Taking the Laplacian on U yields $\Delta V = \operatorname{div}(\nabla V) = \operatorname{div}\vec{F}_{-3} = 0$, which matches the harmonic family $g(r) = \frac{A}{r} + B$ from Question 2(b) with $A = -1$ and $B = C$.

Exercise 2

1. Fix $a > 0$ and consider the two planar curves

$$\gamma_1 : \left[0, \frac{\pi}{2}\right] \to \mathbb{R}^2, \quad \gamma_1(t) = (a\cos^3 t,\, a\sin^3 t), \qquad \gamma_2 : [0,1] \to \mathbb{R}^2, \quad \gamma_2(u) = \left(au^3,\, a(1-u^2)^{3/2}\right).$$



(a) Construct an explicit diffeomorphism $\varphi :]0,1[\to \left]0, \frac{\pi}{2}\right[$ of class C^1 such that $\gamma_2 = \gamma_1 \circ \varphi$ on $]0,1[$,

and state whether φ is orientation-preserving or orientation-reversing. ★★☆☆☆

Method ***Identification of $u = \cos t$ on $\left]0, \frac{\pi}{2}\right[$ and sign analysis of $\varphi'(u)$***

> ***Why we do this (Way of Thinking):***
> Comparing the first coordinates $au^3 = a\cos^3 t$ suggests setting $\cos t = u$. Because cos is strictly decreasing from $\left]0, \frac{\pi}{2}\right[$ onto $]0, 1[$, its inverse $\varphi(u) = \arccos u$ is smooth on the open interval $]0, 1[$ (avoiding the vertical-tangent endpoint $u = 1$), while $\sin(\arccos u) = +\sqrt{1-u^2}$ matches the second coordinate cleanly.

Execution:

- **Step 1 (Definition and C^1-diffeomorphism property of φ):** Define

$$\varphi :]0, 1[\longrightarrow \left]0, \frac{\pi}{2}\right[, \qquad \varphi(u) = \arccos u.$$

On $]0, 1[$, we have $1 - u^2 > 0$, so φ is of class $C^\infty(]0, 1[) \subset C^1(]0, 1[)$ with strictly negative derivative

$$\varphi'(u) = -\frac{1}{\sqrt{1-u^2}} < 0 \qquad \forall u \in]0, 1[.$$

Moreover, φ is bijective from $]0, 1[$ onto $\left]0, \frac{\pi}{2}\right[$ (since $\lim_{u\to 0^+} \arccos u = \frac{\pi}{2}$ and $\lim_{u\to 1^-} \arccos u = 0$), and its inverse $\varphi^{-1} : \left]0, \frac{\pi}{2}\right[\to]0, 1[$ given by $\varphi^{-1}(t) = \cos t$ is also of class C^1 with $(\varphi^{-1})'(t) = -\sin t < 0$. Hence φ is a C^1-diffeomorphism.

- **Step 2 (Reparameterization identity and orientation):** For every $u \in]0, 1[$, $\varphi(u) \in \left]0, \frac{\pi}{2}\right[$ implies $\sin(\varphi(u)) > 0$, so

$$\cos(\varphi(u)) = u, \qquad \sin(\varphi(u)) = \sqrt{1 - \cos^2(\varphi(u))} = \sqrt{1-u^2}.$$

Substituting into γ_1 gives, for every $u \in]0, 1[$:

$$(\gamma_1 \circ \varphi)(u) = \Big(a\big(\cos\varphi(u)\big)^3,\ a\big(\sin\varphi(u)\big)^3\Big) = \Big(au^3,\ a\big(\sqrt{1-u^2}\big)^3\Big) = \big(au^3,\ a(1-u^2)^{3/2}\big) = \gamma_2(u).$$

Because $\varphi'(u) = -\frac{1}{\sqrt{1-u^2}} < 0$ for all $u \in]0, 1[$, φ is **orientation-reversing**: γ_1 traverses the astroid arc from $\gamma_1(0) = (a, 0)$ to $\gamma_1(\frac{\pi}{2}) = (0, a)$, whereas γ_2 traverses the same geometric arc in the opposite direction from $\gamma_2(0) = (0, a)$ to $\gamma_2(1) = (a, 0)$.

$$\boxed{\begin{aligned} &\varphi(u) = \arccos u \quad \text{on }]0, 1[, \\ &\varphi'(u) = -\frac{1}{\sqrt{1-u^2}} < 0 \implies \varphi \text{ is orientation-reversing} \end{aligned}}$$

(b) Determine all regular points of γ_1 on $\left[0, \frac{\pi}{2}\right]$, compute the unit tangent vector $\vec{T}(t)$ and unit normal vector $\vec{N}(t)$ at every regular point, and find the total arc length $\ell(\gamma_1)$. ★★☆☆☆

Method ***Velocity norm factorization $\|\gamma_1'(t)\| = 3a|\sin t \cos t|$ and Frenet planar frame***

> ***Why we do this (Way of Thinking):***
> Differentiating $a\cos^3 t$ and $a\sin^3 t$ extracts a common factor $3a\sin t\cos t$. Isolating where this factor vanishes identifies the singular endpoints, while on the open interval $\left]0, \frac{\pi}{2}\right[$ strict positivity $\sin t \cos t > 0$ removes the absolute value in $\|\gamma_1'(t)\|$ both for normalizing $\vec{T}(t)$ and for integrating the arc length.

Execution:

- **Step 1 (Velocity vector and regular points):** Differentiating $\gamma_1(t) = (a\cos^3 t,\ a\sin^3 t)$ on $\left[0, \frac{\pi}{2}\right]$ by the chain rule gives:

$$\gamma_1'(t) = \big(3a\cos^2 t\,(-\sin t),\ 3a\sin^2 t\,(\cos t)\big) = (-3a\cos^2 t\sin t,\ 3a\sin^2 t\cos t).$$

Computing its Euclidean norm yields:

$$\|\gamma_1'(t)\| = \sqrt{9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t} = \sqrt{9a^2\sin^2 t\cos^2 t\,(\cos^2 t + \sin^2 t)} = 3a\,|\sin t\cos t|.$$

Since $a > 0$, on $\left[0, \frac{\pi}{2}\right]$ we have:

$$\gamma_1'(t) = (0,0) \iff \|\gamma_1'(t)\| = 0 \iff \sin t = 0 \text{ or } \cos t = 0 \iff t \in \left\{0, \frac{\pi}{2}\right\}.$$

Thus $t = 0$ and $t = \frac{\pi}{2}$ are singular points (cusps at $(a,0)$ and $(0,a)$), and the set of regular parameter values is the open interval $\left]0, \frac{\pi}{2}\right[$.

- **Step 2 (Unit tangent $\vec{T}(t)$ and unit normal $\vec{N}(t)$ on $\left]0, \frac{\pi}{2}\right[$):** For every $t \in \left]0, \frac{\pi}{2}\right[$, $\sin t > 0$ and $\cos t > 0$, so $\|\gamma_1'(t)\| = 3a\sin t\cos t > 0$. Dividing $\gamma_1'(t)$ by $\|\gamma_1'(t)\|$ gives:

$$\vec{T}(t) = \frac{\gamma_1'(t)}{\|\gamma_1'(t)\|} = \frac{(-3a\cos^2 t\sin t,\ 3a\sin^2 t\cos t)}{3a\sin t\cos t} = (-\cos t,\ \sin t).$$

Differentiating $\vec{T}(t)$ with respect to t gives the curvature direction vector $\vec{T}'(t) = (\sin t, \cos t)$, which already has unit norm $\sqrt{\sin^2 t + \cos^2 t} = 1$ (equivalently, normalizing the planar normal $\vec{n}(t) = (y_1'(t), -x_1'(t)) = (3a\sin^2 t\cos t,\ 3a\cos^2 t\sin t) = 3a\sin t\cos t\,(\sin t, \cos t)$ yields the same vector):

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} = (\sin t,\ \cos t)$$

(with $(-\sin t, -\cos t)$ being the opposite-orientation unit normal to the planar curve).

- **Step 3 (Total arc length $\ell(\gamma_1)$):** Integrating $\|\gamma_1'(t)\| = 3a\sin t\cos t$ over $\left[0, \frac{\pi}{2}\right]$:

Sub **Antiderivative via $u = \sin t$, $du = \cos t\,dt$, bounds $t = 0 \mapsto u = 0$ and $t = \frac{\pi}{2} \mapsto u = 1$**

$$\ell(\gamma_1) = \int_0^{\pi/2} \|\gamma_1'(t)\|\,dt = 3a\int_0^{\pi/2} \sin t\cos t\,dt = 3a\left[\frac{1}{2}\sin^2 t\right]_0^{\pi/2} = \frac{3a}{2}(1-0) = \frac{3a}{2}.$$

> Regular points: $t \in \left]0, \dfrac{\pi}{2}\right[$, $\quad \vec{T}(t) = (-\cos t,\ \sin t)$,
> $\vec{N}(t) = (\sin t,\ \cos t)$, $\quad \ell(\gamma_1) = \dfrac{3a}{2}$

> ✓ **Sanity Check:**
> The straight chord between $(a,0)$ and $(0,a)$ has length $\sqrt{a^2+a^2} = \sqrt{2}\,a \approx 1.414\,a$, while the two coordinate segments $(a,0) \to (0,0) \to (0,a)$ have combined length $2a$; the convex astroid arc lies strictly between them, consistent with $\sqrt{2}\,a < \frac{3}{2}a = 1.5\,a < 2a$.

2. Consider the piecewise-defined space curve $\gamma : \mathbb{R} \to \mathbb{R}^3$ given by

$$\gamma(t) = \begin{cases} \big((a^2 - 2a)t,\ bt + t^2,\ 0\big), & \text{if } t < 0, \\ \big(\sin(at),\ \sin((3b - b^2)t),\ t^3\big), & \text{if } t \geq 0. \end{cases}$$

Determine all pairs $(a,b)\in\mathbb{R}^2$ such that γ is of class C^1 on $\mathbb{R}$ and regular at every $t\in\mathbb{R}$. ★★★☆☆

Method ***Junction C^1 matching at $t=0$ followed by global non-vanishing test $\gamma'(t)\neq\vec{0}$ on $\mathbb{R}$***

> ***Why we do this (Way of Thinking):***
> Both branches of γ are smooth away from $t=0$, so C^1 smoothness on $\mathbb{R}$ reduces to continuity at $t=0$ together with equality of the one-sided derivative limits $\lim_{t\to0^-}\gamma'(t)=\lim_{t\to0^+}\gamma'(t)$. However, global regularity demands $\gamma'(t)\neq(0,0,0)$ at *every* $t\in\mathbb{R}$, not only at the junction $t=0$: each candidate (a,b) produced by the C^1 condition must be tested on all three regions $t<0$, $t=0$, and $t>0$.

Execution:

- **Step 1 (Continuity on $\mathbb{R}$):** On $]-\infty,0[$ and $]0,+\infty[$, the components of γ are polynomial and trigonometric functions, hence of class C^∞. At $t=0$, we have $\gamma(0)=(\sin 0,\ \sin 0,\ 0^3)=(0,0,0)$ and

$$\lim_{t\to0^-}\gamma(t)=\lim_{t\to0^-}\big((a^2-2a)t,\ bt+t^2,\ 0\big)=(0,0,0),$$
$$\lim_{t\to0^+}\gamma(t)=\lim_{t\to0^+}\big(\sin(at),\ \sin((3b-b^2)t),\ t^3\big)=(0,0,0).$$

  Hence γ is continuous on $\mathbb{R}$ for every $(a,b)\in\mathbb{R}^2$.

- **Step 2 (C^1 condition at $t=0$):** Differentiating each smooth branch away from $t=0$ gives:

$$\gamma'(t)=\begin{cases}(a^2-2a,\ b+2t,\ 0), & \text{if } t<0,\\ \big(a\cos(at),\ (3b-b^2)\cos((3b-b^2)t),\ 3t^2\big), & \text{if } t>0.\end{cases}$$

  Since γ is continuous at $t=0$ and C^1 on $\mathbb{R}\setminus\{0\}$, γ belongs to $C^1(\mathbb{R},\mathbb{R}^3)$ if and only if $\lim_{t\to0^-}\gamma'(t)=\lim_{t\to0^+}\gamma'(t)$, in which case their common limit equals $\gamma'(0)$. Evaluating both one-sided limits:

$$\lim_{t\to0^-}\gamma'(t)=(a^2-2a,\ b,\ 0),\qquad \lim_{t\to0^+}\gamma'(t)=\big(a\cos 0,\ (3b-b^2)\cos 0,\ 0\big)=(a,\ 3b-b^2,\ 0).$$

  Equating the corresponding coordinates yields:

$$\begin{cases}a^2-2a=a,\\ b=3b-b^2,\\ 0=0\end{cases}\iff\begin{cases}a^2-3a=a(a-3)=0,\\ b^2-2b=b(b-2)=0\end{cases}\iff a\in\{0,3\}\text{ and }b\in\{0,2\}.$$

  Thus $\gamma\in C^1(\mathbb{R},\mathbb{R}^3)$ if and only if $(a,b)\in\{(0,0),\ (0,2),\ (3,0),\ (3,2)\}$.

- **Step 3 (Global regularity $\gamma'(t)\neq(0,0,0)$ for all $t\in\mathbb{R}$):** We inspect $\gamma'(t)$ on $]-\infty,0[$, $\{0\}$, and $]0,+\infty[$ for each of the four C^1 candidates:
  - **Case $(a,b)=(0,0)$:** At $t=0$, $\gamma'(0)=(a,\ 3b-b^2,\ 0)=(0,0,0)$, so γ is singular at $t=0$. *Rejected.*
  - **Case $(a,b)=(0,2)$:** For $t<0$, substituting $a=0$ and $b=2$ gives

$$\gamma'(t)=(0^2-0,\ 2+2t,\ 0)=\big(0,\ 2(1+t),\ 0\big).$$

    At $t=-1\in]-\infty,0[$, we obtain $\gamma'(-1)=(0,0,0)$, so γ fails to be regular at $t=-1$. *Rejected.*
  - **Case $(a,b)=(3,0)$:** For $t<0$, $\gamma'(t)=(3,\ 2t,\ 0)$ has first coordinate $3\neq0$; at $t=0$, $\gamma'(0)=(3,0,0)\neq(0,0,0)$; for $t>0$, $\gamma'(t)=(3\cos(3t),\ 0,\ 3t^2)$ has third coordinate $3t^2>0$. Hence $\gamma'(t)\neq(0,0,0)$ for every $t\in\mathbb{R}$. *Accepted.*
  - **Case $(a,b)=(3,2)$:** For $t<0$, $\gamma'(t)=(3,\ 2+2t,\ 0)$ has first coordinate $3\neq0$; at $t=0$, $\gamma'(0)=(3,2,0)\neq(0,0,0)$; for $t>0$, $\gamma'(t)=(3\cos(3t),\ 2\cos(2t),\ 3t^2)$ has third coordinate $3t^2>0$. Hence $\gamma'(t)\neq(0,0,0)$ for every $t\in\mathbb{R}$. *Accepted.*

$$\boxed{(a,b)\in\big\{(3,0),\ (3,2)\big\}}$$

> ► **Common Pitfall:**
> Checking $\gamma'(0) \neq (0,0,0)$ only at the gluing point $t = 0$ eliminates $(0,0)$ while mistakenly keeping $(0,2)$, since $\gamma'(0) = (0,2,0) \neq (0,0,0)$ when $(a,b) = (0,2)$. Whenever $a = 0$, the first and third coordinates of $\gamma'(t)$ vanish identically on $]-\infty, 0[$, leaving $\gamma'(t) = (0, b+2t, 0)$, which vanishes at $t = -\frac{b}{2} = -1 < 0$ for $b = 2$.

3. Determine all $k \in \mathbb{R}$ such that the arc length of $\gamma(t) = \left(2t,\ kt^2,\ \frac{1}{3}t^3\right)$ on $[0,T]$ equals $2T + \frac{1}{3}T^3$ for every $T > 0$. ★★☆☆☆

Method *Fundamental Theorem of Calculus differentiation with respect to the upper bound T*

> ***Why we do this (Way of Thinking):***
> Instead of attempting to evaluate the elliptic-type integral $\int_0^T \sqrt{4+4k^2t^2+t^4}\,dt$ directly for arbitrary k, differentiating the identity $L(T) = 2T + \frac{1}{3}T^3$ with respect to $T > 0$ immediately equates the continuous integrand $\|\gamma'(T)\|$ to $2 + T^2$, turning the integral condition into a polynomial identity in T.

Execution:

- **Step 1 (Velocity norm and arc-length function):** Differentiating $\gamma(t) = \left(2t,\ kt^2,\ \frac{1}{3}t^3\right)$ gives

$$\gamma'(t) = (2,\ 2kt,\ t^2), \qquad \|\gamma'(t)\| = \sqrt{2^2 + (2kt)^2 + (t^2)^2} = \sqrt{4+4k^2t^2+t^4}.$$

  The arc length of γ on $[0,T]$ is therefore $L(T) = \int_0^T \sqrt{4+4k^2t^2+t^4}\,dt$.

- **Step 2 (Necessary and sufficient condition via differentiation in T):** Since $t \mapsto \sqrt{4+4k^2t^2+t^4}$ is continuous on $[0,+\infty[$, L is continuous at $T = 0$ with $L(0) = 0 = 2(0) + \frac{1}{3}(0)^3$ and differentiable on $]0,+\infty[$ by the Fundamental Theorem of Calculus, with $L'(T) = \sqrt{4+4k^2T^2+T^4}$. Because two differentiable functions on $[0,+\infty[$ with the same value at 0 coincide on $]0,+\infty[$ if and only if their derivatives coincide on $]0,+\infty[$:

$$L(T) = 2T + \frac{1}{3}T^3 \quad \forall T > 0 \iff \sqrt{4+4k^2T^2+T^4} = \frac{d}{dT}\left(2T + \frac{1}{3}T^3\right) = 2 + T^2 \quad \forall T > 0.$$

  Since both sides are strictly positive for all $T > 0$, squaring preserves equivalence:

$$4 + 4k^2T^2 + T^4 = (2+T^2)^2 = 4 + 4T^2 + T^4 \iff 4(k^2-1)T^2 = 0 \quad \forall T > 0.$$

  Dividing by $4T^2 > 0$ gives $k^2 - 1 = 0 \iff k \in \{-1, 1\}$. Conversely, when $k = \pm 1$, we have $k^2 = 1$, $\|\gamma'(t)\| = \sqrt{(2+t^2)^2} = 2 + t^2$, and $\int_0^T (2+t^2)\,dt = \left[2t + \frac{1}{3}t^3\right]_0^T = 2T + \frac{1}{3}T^3$ for every $T > 0$.

$$\boxed{k \in \{-1,\ 1\}}$$

4. Express the arc length element $ds = \|\gamma'(t)\|\,dt$ of a C^1 space curve given in cylindrical coordinates by $x(t) = r(t)\cos\theta(t)$, $y(t) = r(t)\sin\theta(t)$, $z(t) = z(t)$ in terms of r, θ, z, and their derivatives. Compute the length of the conical spiral $r(t) = e^t$, $\theta(t) = \sqrt{2}\,t$, $z(t) = e^t$ for $t \in [0, \ln 3]$. ★★☆☆☆

Method *Orthogonal decomposition of $(x'(t), y'(t))$ into radial and azimuthal components*

> ***Why we do this (Way of Thinking):***
> Differentiating $x = r\cos\theta$ and $y = r\sin\theta$ expresses (x', y') along the orthogonal unit basis $\vec{e}_r = (\cos\theta, \sin\theta)$ and $\vec{e}_\theta = (-\sin\theta, \cos\theta)$. Expanding $x'^2 + y'^2$ cancels the cross product $2rr'\theta'\sin\theta\cos\theta$ and factors $\cos^2\theta + \sin^2\theta = 1$, reducing the speed of the conical spiral to an exact multiple of e^t.

Execution:

- **Step 1 (Cylindrical arc-length element ds):** Applying the product and chain rules to $x(t) = r(t)\cos\theta(t)$ and $y(t) = r(t)\sin\theta(t)$ gives:

$$x'(t) = r'(t)\cos\theta(t) - r(t)\theta'(t)\sin\theta(t), \qquad y'(t) = r'(t)\sin\theta(t) + r(t)\theta'(t)\cos\theta(t).$$

Squaring each derivative explicitly yields:

$$x'(t)^2 = r'(t)^2\cos^2\theta(t) - 2r(t)r'(t)\theta'(t)\sin\theta(t)\cos\theta(t) + r(t)^2\theta'(t)^2\sin^2\theta(t),$$
$$y'(t)^2 = r'(t)^2\sin^2\theta(t) + 2r(t)r'(t)\theta'(t)\sin\theta(t)\cos\theta(t) + r(t)^2\theta'(t)^2\cos^2\theta(t).$$

Adding the two equations cancels the opposite middle terms $\mp 2r(t)r'(t)\theta'(t)\sin\theta(t)\cos\theta(t)$ and groups $\cos^2\theta(t) + \sin^2\theta(t) = 1$:

$$x'(t)^2 + y'(t)^2 = r'(t)^2\big(\cos^2\theta(t) + \sin^2\theta(t)\big) + r(t)^2\theta'(t)^2\big(\sin^2\theta(t) + \cos^2\theta(t)\big) = r'(t)^2 + r(t)^2\theta'(t)^2.$$

Adding $z'(t)^2$ gives $\|\gamma'(t)\| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} = \sqrt{r'(t)^2 + r(t)^2\theta'(t)^2 + z'(t)^2}$, whence

$$ds = \|\gamma'(t)\|\,dt = \sqrt{r'(t)^2 + r(t)^2\theta'(t)^2 + z'(t)^2}\,dt.$$

- **Step 2 (Length of the conical spiral on $[0, \ln 3]$):** For $r(t) = e^t$, $\theta(t) = \sqrt{2}\,t$, and $z(t) = e^t$, differentiating gives:

$$r'(t) = e^t, \qquad \theta'(t) = \sqrt{2}, \qquad z'(t) = e^t.$$

Substituting into the cylindrical speed formula and noting that $e^t > 0$ for all $t \in \mathbb{R}$:

$$\|\gamma'(t)\| = \sqrt{(e^t)^2 + (e^t)^2(\sqrt{2})^2 + (e^t)^2} = \sqrt{e^{2t} + 2e^{2t} + e^{2t}} = \sqrt{4e^{2t}} = 2e^t.$$

Integrating over $t \in [0, \ln 3]$ gives the arc length:

$$\ell(\gamma) = \int_0^{\ln 3} 2e^t\,dt = \big[2e^t\big]_0^{\ln 3} = 2e^{\ln 3} - 2e^0 = 2(3-1) = 4.$$

$$\boxed{ds = \sqrt{r'(t)^2 + r(t)^2\theta'(t)^2 + z'(t)^2}\,dt, \qquad \ell(\gamma) = 4}$$

Exercise 3

1. Consider the differential 1-form on $\mathbb{R}^3$ defined by

$$\omega = \big(2xye^z + \alpha z^2\big)\,dx + \big(x^2e^z + 3y^2z\big)\,dy + \big(\beta x^2ye^z + 2xz + y^3\big)\,dz, \qquad (\alpha, \beta) \in \mathbb{R}^2.$$

Determine (α, β) so that ω is exact on $\mathbb{R}^3$, and find all functions $f \in C^1(\mathbb{R}^3, \mathbb{R})$ satisfying $df = \omega$.

★★☆☆☆

Method *Poincaré's Theorem on Convex Domains and Sequential Partial Integration*

> ***Why we do this (Way of Thinking):***
> To characterize exactness of a smooth differential 1-form $\omega = P\,dx + Q\,dy + R\,dz$ on $\mathbb{R}^3$, we combine coefficient regularity with domain topology. Because $\mathbb{R}^3$ is convex (and therefore star-shaped), Poincaré's theorem guarantees that ω is exact on $\mathbb{R}^3$ if and only if ω is closed on $\mathbb{R}^3$, meaning the three pairs of cross partial derivatives match identically across $\mathbb{R}^3$. Enforcing $\partial_y P = \partial_x Q$, $\partial_z Q = \partial_y R$, and $\partial_z P = \partial_x R$ fixes (α, β); integrating $\partial_x f = P$ with respect to x up to an arbitrary smooth function $\Phi(y, z)$ and differentiating with respect to y and z then recovers every primitive on the connected space $\mathbb{R}^3$.

Execution: Write $\omega = P\,dx + Q\,dy + R\,dz$ with

$$P(x,y,z) = 2xye^z + \alpha z^2, \qquad Q(x,y,z) = x^2e^z + 3y^2z, \qquad R(x,y,z) = \beta x^2ye^z + 2xz + y^3.$$

Each component is a sum of products of polynomials and e^z, so $P, Q, R \in C^\infty(\mathbb{R}^3, \mathbb{R})$. Since $\mathbb{R}^3$ is convex and hence star-shaped, Poincaré's theorem (together with Schwarz's theorem for necessity) states that ω is exact on $\mathbb{R}^3$ if and only if ω is closed on $\mathbb{R}^3$:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \qquad \text{for all } (x, y, z) \in \mathbb{R}^3.$$

We compute all six cross partial derivatives explicitly:

- **First pair** (x, y)**:**

$$\frac{\partial P}{\partial y}(x, y, z) = \frac{\partial}{\partial y}\left(2xye^z + \alpha z^2\right) = 2xe^z, \qquad \frac{\partial Q}{\partial x}(x, y, z) = \frac{\partial}{\partial x}\left(x^2e^z + 3y^2z\right) = 2xe^z.$$

  Hence $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ holds identically on $\mathbb{R}^3$ for all $(\alpha, \beta) \in \mathbb{R}^2$.

- **Second pair** (y, z)**:**

$$\frac{\partial Q}{\partial z}(x, y, z) = \frac{\partial}{\partial z}\left(x^2e^z + 3y^2z\right) = x^2e^z + 3y^2, \qquad \frac{\partial R}{\partial y}(x, y, z) = \frac{\partial}{\partial y}\left(\beta x^2ye^z + 2xz + y^3\right) = \beta x^2e^z + 3y^2.$$

  Equating both expressions across $\mathbb{R}^3$ yields

$$x^2e^z + 3y^2 = \beta x^2e^z + 3y^2 \iff (\beta - 1)x^2e^z = 0 \qquad \forall (x, y, z) \in \mathbb{R}^3.$$

  Evaluating at $(x, y, z) = (1, 0, 0)$ gives $(\beta - 1) \cdot 1 \cdot e^0 = 0$, so $\beta = 1$; conversely, $\beta = 1$ makes $(\beta - 1)x^2e^z = 0$ on all of $\mathbb{R}^3$.

- **Third pair** (x, z)**:**

$$\frac{\partial P}{\partial z}(x, y, z) = \frac{\partial}{\partial z}\left(2xye^z + \alpha z^2\right) = 2xye^z + 2\alpha z, \qquad \frac{\partial R}{\partial x}(x, y, z) = \frac{\partial}{\partial x}\left(\beta x^2ye^z + 2xz + y^3\right) = 2\beta xye^z + 2z.$$

  Substituting $\beta = 1$ and equating both derivatives on $\mathbb{R}^3$ gives

$$2xye^z + 2\alpha z = 2xye^z + 2z \iff 2(\alpha - 1)z = 0 \qquad \forall (x, y, z) \in \mathbb{R}^3.$$

  Evaluating at $(x, y, z) = (0, 0, 1)$ gives $2(\alpha - 1) = 0$, so $\alpha = 1$; conversely, $(\alpha, \beta) = (1, 1)$ satisfies all three closedness equations everywhere on $\mathbb{R}^3$.

For $(\alpha, \beta) = (1, 1)$, a function $f \in C^1(\mathbb{R}^3, \mathbb{R})$ satisfies $df = \omega$ on $\mathbb{R}^3$ if and only if

$$\text{(i) } \frac{\partial f}{\partial x}(x, y, z) = 2xye^z + z^2, \qquad \text{(ii) } \frac{\partial f}{\partial y}(x, y, z) = x^2e^z + 3y^2z, \qquad \text{(iii) } \frac{\partial f}{\partial z}(x, y, z) = x^2ye^z + 2xz + y^3.$$

Step 1 — Integrate (i) with respect to x**:** Holding $(y, z) \in \mathbb{R}^2$ fixed,

$$f(x, y, z) = \int \left(2xye^z + z^2\right) dx = x^2ye^z + xz^2 + \Phi(y, z),$$

where $\Phi \in C^1(\mathbb{R}^2, \mathbb{R})$ is an arbitrary C^1 function of (y, z).

Step 2 — Differentiate with respect to y **and match (ii):**

$$\frac{\partial f}{\partial y}(x, y, z) = x^2e^z + \frac{\partial \Phi}{\partial y}(y, z) = x^2e^z + 3y^2z \iff \frac{\partial \Phi}{\partial y}(y, z) = 3y^2z.$$

Holding $z \in \mathbb{R}$ fixed and integrating with respect to y yields

$$\Phi(y, z) = \int 3y^2z \, dy = y^3z + \Psi(z),$$

where $\Psi \in C^1(\mathbb{R}, \mathbb{R})$ depends only on z. Substituting back gives $f(x, y, z) = x^2ye^z + xz^2 + y^3z + \Psi(z)$.

Step 3 — Differentiate with respect to z and match (iii):

$$\frac{\partial f}{\partial z}(x,y,z) = x^2ye^z + 2xz + y^3 + \Psi'(z) = x^2ye^z + 2xz + y^3 \iff \Psi'(z) = 0 \qquad \forall z \in \mathbb{R}.$$

Because $\mathbb{R}$ is an interval, $\Psi(z) = C$ for an arbitrary constant $C \in \mathbb{R}$.

$$\boxed{\begin{aligned}&(\alpha,\beta) = (1,1),\\ &f(x,y,z) = x^2ye^z + xz^2 + y^3z + C, \quad C \in \mathbb{R}\end{aligned}}$$

> ► **Common Pitfall:**
> Writing a scalar constant $+C$ immediately after integrating $\partial_x f = P$ instead of a two-variable function $+\Phi(y,z)$ drops the purely (y,z)-dependent term y^3z, which vanishes under ∂_x and would render the remaining equations inconsistent.

> ✓ **Sanity Check:**
> Taking the differential of $f(x,y,z) = x^2ye^z + xz^2 + y^3z + C$ yields $df = (2xye^z + z^2)\,dx + (x^2e^z + 3y^2z)\,dy + (x^2ye^z + 2xz + y^3)\,dz$, which coincides with ω when $(\alpha,\beta) = (1,1)$.

2. Let $\varphi, \psi \in C^1(\mathbb{R},\mathbb{R})$ satisfy $\psi(0) = 1$, and consider the vector field on $\mathbb{R}^3$ given by

$$\vec{V}(x,y,z) = yz\,\varphi(x)\,\vec{i} + 2z\,\psi(x)\,\vec{j} + \Big(y\big(\varphi(x) + \psi(x)\big) + 2z\Big)\vec{k}.$$

(a) Compute $\operatorname{div}\vec{V}$ **and** $\operatorname{rot}\vec{V}$ **in terms of** φ**,** ψ**, and their derivatives.** ★★☆☆☆

Method *Cartesian Divergence and Curl Operators on Factorized Components*

> ***Why we do this (Way of Thinking):***
> Denote the Cartesian components of $\vec{V}$ by $V_1(x,y,z) = yz\,\varphi(x)$, $V_2(x,y,z) = 2z\,\psi(x)$, and $V_3(x,y,z) = y\big(\varphi(x) + \psi(x)\big) + 2z$. Because φ and ψ are functions of x alone, partial differentiation with respect to y or z treats $\varphi(x)$ and $\psi(x)$ as constants, while partial differentiation with respect to x differentiates $\varphi(x)$ and $\psi(x)$ directly.

Execution: For the divergence $\operatorname{div}\vec{V} = \dfrac{\partial V_1}{\partial x} + \dfrac{\partial V_2}{\partial y} + \dfrac{\partial V_3}{\partial z}$, compute the three diagonal partial derivatives:

$$\frac{\partial V_1}{\partial x} = \frac{\partial}{\partial x}\big(yz\,\varphi(x)\big) = yz\,\varphi'(x), \quad \frac{\partial V_2}{\partial y} = \frac{\partial}{\partial y}\big(2z\,\psi(x)\big) = 0, \quad \frac{\partial V_3}{\partial z} = \frac{\partial}{\partial z}\Big(y\big(\varphi(x) + \psi(x)\big) + 2z\Big) = 2.$$

Adding the three terms gives $\operatorname{div}\vec{V}(x,y,z) = yz\,\varphi'(x) + 2$.
For the curl $\operatorname{rot}\vec{V} = \nabla \times \vec{V}$, compute all six off-diagonal first-order partial derivatives:

$$\frac{\partial V_3}{\partial y} = \frac{\partial}{\partial y}\Big(y\big(\varphi(x) + \psi(x)\big) + 2z\Big) = \varphi(x) + \psi(x), \qquad \frac{\partial V_2}{\partial z} = \frac{\partial}{\partial z}\big(2z\,\psi(x)\big) = 2\psi(x),$$

$$\frac{\partial V_1}{\partial z} = \frac{\partial}{\partial z}\big(yz\,\varphi(x)\big) = y\,\varphi(x), \qquad \frac{\partial V_3}{\partial x} = \frac{\partial}{\partial x}\Big(y\big(\varphi(x) + \psi(x)\big) + 2z\Big) = y\big(\varphi'(x) + \psi'(x)\big),$$

$$\frac{\partial V_2}{\partial x} = \frac{\partial}{\partial x}\big(2z\,\psi(x)\big) = 2z\,\psi'(x), \qquad \frac{\partial V_1}{\partial y} = \frac{\partial}{\partial y}\big(yz\,\varphi(x)\big) = z\,\varphi(x).$$

Forming the three curl components yields

$$(\operatorname{rot}\vec{V})_x = \frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} = \big(\varphi(x) + \psi(x)\big) - 2\psi(x) = \varphi(x) - \psi(x),$$

$$(\operatorname{rot}\vec{V})_y = \frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x} = y\,\varphi(x) - y\big(\varphi'(x) + \psi'(x)\big) = y\big(\varphi(x) - \varphi'(x) - \psi'(x)\big),$$

$$(\operatorname{rot}\vec{V})_z = \frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} = 2z\,\psi'(x) - z\,\varphi(x) = z\big(2\psi'(x) - \varphi(x)\big).$$

$$\operatorname{div}\vec{V}(x,y,z) = yz\,\varphi'(x) + 2,$$
$$\operatorname{rot}\vec{V}(x,y,z) = \big(\varphi(x) - \psi(x)\big)\,\vec{i} + y\big(\varphi(x) - \varphi'(x) - \psi'(x)\big)\,\vec{j} + z\big(2\psi'(x) - \varphi(x)\big)\,\vec{k}$$

► **Common Pitfall:**
In the middle component of $\operatorname{rot}\vec{V}$, the cofactor sign produces $\partial_z V_1 - \partial_x V_3$ rather than $\partial_x V_3 - \partial_z V_1$; reversing this sign swaps the signs inside $y\big(\varphi(x) - \varphi'(x) - \psi'(x)\big)$.

(b) Determine φ and ψ such that $\vec{V}$ is conservative on $\mathbb{R}^3$, and find all scalar potentials of $\vec{V}$ on $\mathbb{R}^3$.
★★★☆☆

Method ***Irrotationality on $\mathbb{R}^3$, Integrating-Factor ODE Solution, and Potential Recovery***

Why we do this (Way of Thinking):
Since $\varphi, \psi \in C^1(\mathbb{R},\mathbb{R})$, the field $\vec{V}$ belongs to $C^1(\mathbb{R}^3,\mathbb{R}^3)$. Because $\mathbb{R}^3$ is convex (hence star-shaped), Poincaré's theorem implies that $\vec{V}$ is conservative on $\mathbb{R}^3$ if and only if $\operatorname{rot}\vec{V} = \vec{0}$ identically on $\mathbb{R}^3$. Equating each component of $\operatorname{rot}\vec{V}$ from part (a) to zero for all $(x,y,z) \in \mathbb{R}^3$ reduces the problem to a first-order linear ODE in $\psi(x)$ anchored by $\psi(0) = 1$, after which sequential integration yields all scalar potentials h satisfying $\nabla h = \vec{V}$.

Execution: By Poincaré's theorem on the star-shaped domain $\mathbb{R}^3$ for $\vec{V} \in C^1(\mathbb{R}^3,\mathbb{R}^3)$, $\vec{V}$ is conservative on $\mathbb{R}^3$ if and only if $\operatorname{rot}\vec{V}(x,y,z) = \vec{0}$ for every $(x,y,z) \in \mathbb{R}^3$:

$$\begin{cases} \varphi(x) - \psi(x) = 0, \\ y\big(\varphi(x) - \varphi'(x) - \psi'(x)\big) = 0, \\ z\big(2\psi'(x) - \varphi(x)\big) = 0, \end{cases} \qquad \forall (x,y,z) \in \mathbb{R}^3.$$

The first equation requires $\varphi(x) = \psi(x)$ for every $x \in \mathbb{R}$. Evaluating the third equation at $z = 1$ gives $2\psi'(x) - \varphi(x) = 0$ for all $x \in \mathbb{R}$; substituting $\varphi(x) = \psi(x)$ produces the homogeneous linear ODE

$$2\psi'(x) - \psi(x) = 0 \iff \psi'(x) - \frac{1}{2}\psi(x) = 0 \qquad \forall x \in \mathbb{R}.$$

When $\varphi = \psi$ and $2\psi' = \psi$, we also have $\varphi'(x) = \psi'(x)$, so the second equation becomes

$$y\big(\varphi(x) - \varphi'(x) - \psi'(x)\big) = y\big(\psi(x) - 2\psi'(x)\big) = 0 \qquad \forall (x,y,z) \in \mathbb{R}^3,$$

which is automatically satisfied everywhere on $\mathbb{R}^3$.
Multiplying $\psi'(x) - \frac{1}{2}\psi(x) = 0$ by the non-vanishing integrating factor $e^{-x/2}$ yields

$$e^{-x/2}\psi'(x) - \frac{1}{2}e^{-x/2}\psi(x) = 0 \iff \frac{d}{dx}\Big(e^{-x/2}\psi(x)\Big) = 0 \qquad \forall x \in \mathbb{R}.$$

Integrating over the interval $\mathbb{R}$ gives $e^{-x/2}\psi(x) = K$ for some constant $K \in \mathbb{R}$, that is, $\psi(x) = Ke^{x/2}$. Imposing $\psi(0) = 1$ gives $Ke^0 = 1$, hence $K = 1$, and therefore

$$\varphi(x) = \psi(x) = e^{x/2} \qquad \forall x \in \mathbb{R}.$$

With $\varphi(x) = \psi(x) = e^{x/2}$, the vector field takes the explicit form

$$\vec{V}(x,y,z) = yz\,e^{x/2}\,\vec{i} + 2z\,e^{x/2}\,\vec{j} + \big(2y\,e^{x/2} + 2z\big)\,\vec{k}.$$

We now solve $\nabla h = \vec{V}$ for $h \in C^1(\mathbb{R}^3,\mathbb{R})$:

$$\frac{\partial h}{\partial x} = yz\,e^{x/2}, \qquad \frac{\partial h}{\partial y} = 2z\,e^{x/2}, \qquad \frac{\partial h}{\partial z} = 2y\,e^{x/2} + 2z.$$

Integrating $\partial_x h = yz\,e^{x/2}$ with respect to x at fixed (y,z) gives

$$h(x,y,z) = \int yz\,e^{x/2}\,dx = 2yz\,e^{x/2} + \Phi(y,z), \qquad \Phi \in C^1(\mathbb{R}^2,\mathbb{R}).$$

Differentiating with respect to y and equating to $\partial_y h = 2z\,e^{x/2}$ yields

$$2z\,e^{x/2} + \frac{\partial \Phi}{\partial y}(y,z) = 2z\,e^{x/2} \iff \frac{\partial \Phi}{\partial y}(y,z) = 0 \implies \Phi(y,z) = \Psi(z),$$

for some $\Psi \in C^1(\mathbb{R},\mathbb{R})$. Differentiating $h(x,y,z) = 2yz\,e^{x/2} + \Psi(z)$ with respect to z and matching $\partial_z h = 2y\,e^{x/2} + 2z$ gives

$$2y\,e^{x/2} + \Psi'(z) = 2y\,e^{x/2} + 2z \iff \Psi'(z) = 2z \implies \Psi(z) = z^2 + C, \quad C \in \mathbb{R}.$$

$$\begin{aligned} \varphi(x) &= \psi(x) = e^{x/2}, \\ h(x,y,z) &= 2yz\,e^{x/2} + z^2 + C, \quad C \in \mathbb{R} \end{aligned}$$

✓ **Sanity Check:**
Differentiating $h(x,y,z) = 2yz\,e^{x/2} + z^2 + C$ confirms $\nabla h = \left(yz\,e^{x/2},\ 2z\,e^{x/2},\ 2y\,e^{x/2} + 2z\right) = \vec{V}(x,y,z)$, while $\operatorname{div}\vec{V} = \Delta h = \frac{1}{2}yz\,e^{x/2} + 0 + 2 = yz\,\varphi'(x) + 2$, matching part (a).

3. On the open half-plane $U = \{(x,y) \in \mathbb{R}^2 : y > 0\}$, consider $\vec{F}(x,y) = 2xy\,\vec{i} + (y^2 - 2x^2)\,\vec{j}$.

(a) Verify that $\vec{F}$ is not conservative on U. ★☆☆☆☆

Method *Contrapositive of Schwarz's Symmetry Theorem*

Why we do this (Way of Thinking):
If a smooth planar vector field $\vec{F} = P\,\vec{i} + Q\,\vec{j}$ admitted a scalar potential f on U, then f would be C^2 on U and Schwarz's theorem would require $\partial_y P = \partial_x Q$ at every point of U. Computing $\partial_x Q - \partial_y P$ and finding a point in U where this difference is non-zero rules out the existence of a potential on U.

Execution: Set $P(x,y) = 2xy$ and $Q(x,y) = y^2 - 2x^2$, which belong to $C^\infty(U,\mathbb{R})$. On U,

$$\frac{\partial P}{\partial y}(x,y) = \frac{\partial}{\partial y}(2xy) = 2x, \qquad \frac{\partial Q}{\partial x}(x,y) = \frac{\partial}{\partial x}(y^2 - 2x^2) = -4x.$$

Consequently,

$$\frac{\partial Q}{\partial x}(x,y) - \frac{\partial P}{\partial y}(x,y) = -4x - 2x = -6x.$$

At $(x,y) = (1,1) \in U$ (since $y = 1 > 0$), we have $\dfrac{\partial Q}{\partial x}(1,1) - \dfrac{\partial P}{\partial y}(1,1) = -6 \neq 0$. By the contrapositive of Schwarz's theorem, no function $f \in C^1(U,\mathbb{R})$ satisfies $\nabla f = \vec{F}$ on U.

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -6x \not\equiv 0 \text{ on } U \implies \vec{F} \text{ is not conservative on } U$$

(b) Find the unique function $\mu \in C^1(]0,+\infty[,\mathbb{R}^*)$ with $\mu(1) = 1$ such that $\mu(y)\,\vec{F}(x,y)$ is conservative

on U, and determine all scalar potentials of $\mu(y)\,\vec{F}(x,y)$ on U. ★★★☆☆

Method ***Single-Variable Integrating Factor $\mu(y)$ and Poincaré's Theorem on Convex Half-Planes***

> ***Why we do this (Way of Thinking):***
> Scaling $\vec{F}$ by a non-zero factor $\mu(y)$ produces $\vec{G}(x,y) = 2xy\,\mu(y)\,\vec{\imath} + (y^2 - 2x^2)\,\mu(y)\,\vec{\jmath}$. Because the open upper half-plane $U = \mathbb{R}\times]0,+\infty[$ is convex (and therefore star-shaped), Poincaré's theorem guarantees that $\vec{G}$ is conservative on U if and only if $\partial_y\big(2xy\,\mu(y)\big) = \partial_x\big((y^2-2x^2)\,\mu(y)\big)$ throughout U. Expanding the y-derivative by the product rule cancels the x-dependence and isolates a separable first-order ODE for $\mu(y)$ on $]0,+\infty[$.

Execution: Let $\mu \in C^1(]0,+\infty[,\mathbb{R}^*)$ satisfy $\mu(1) = 1$, and set $\vec{G}(x,y) = \mu(y)\,\vec{F}(x,y) = \widetilde{P}(x,y)\,\vec{\imath} + \widetilde{Q}(x,y)\,\vec{\jmath}$ on U, where

$$\widetilde{P}(x,y) = 2xy\,\mu(y), \qquad \widetilde{Q}(x,y) = (y^2 - 2x^2)\,\mu(y).$$

Since $\mu \in C^1(]0,+\infty[)$, we have $\vec{G} \in C^1(U,\mathbb{R}^2)$. Moreover, for any $(x_1,y_1),(x_2,y_2) \in U$ and $t \in [0,1]$, we have $(1-t)y_1 + ty_2 > 0$, so U is convex and hence star-shaped. By Poincaré's theorem, $\vec{G}$ is conservative on U if and only if $\dfrac{\partial \widetilde{P}}{\partial y} = \dfrac{\partial \widetilde{Q}}{\partial x}$ on U.

Applying the product rule in y to $\widetilde{P}$ and differentiating $\widetilde{Q}$ with respect to x gives

$$\frac{\partial \widetilde{P}}{\partial y}(x,y) = 2x\,\mu(y) + 2xy\,\mu'(y), \qquad \frac{\partial \widetilde{Q}}{\partial x}(x,y) = -4x\,\mu(y).$$

Equating both partial derivatives across U yields

$$2x\,\mu(y) + 2xy\,\mu'(y) = -4x\,\mu(y) \iff 2x\big(y\,\mu'(y) + 3\,\mu(y)\big) = 0 \qquad \forall (x,y) \in U.$$

Setting $x = 1$ gives $2\big(y\,\mu'(y) + 3\,\mu(y)\big) = 0$ for every $y > 0$, and conversely this ODE implies $2x\big(y\,\mu'(y) + 3\,\mu(y)\big) = 0$ for all $(x,y) \in U$. Hence $\vec{G}$ is conservative on U if and only if

$$y\,\mu'(y) + 3\,\mu(y) = 0 \qquad \forall y \in]0,+\infty[.$$

Because μ is continuous on the interval $]0,+\infty[$, never vanishes on $]0,+\infty[$, and satisfies $\mu(1) = 1 > 0$, the intermediate value theorem forces $\mu(y) > 0$ for all $y > 0$. Dividing by $y\,\mu(y) > 0$ gives

$$\frac{\mu'(y)}{\mu(y)} = -\frac{3}{y} \qquad \forall y > 0.$$

Integrating on $]0,+\infty[$ yields

$$\ln \mu(y) = -3\ln y + C_0 = \ln(y^{-3}) + C_0 \implies \mu(y) = Ky^{-3} \qquad (K = e^{C_0} > 0).$$

Imposing $\mu(1) = 1$ gives $K \cdot 1^{-3} = 1$, so $K = 1$ and uniquely $\mu(y) = \dfrac{1}{y^3}$ for all $y > 0$.

Substituting $\mu(y) = y^{-3}$ into $\vec{G}$ gives

$$\vec{G}(x,y) = \frac{2x}{y^2}\,\vec{\imath} + \left(\frac{1}{y} - \frac{2x^2}{y^3}\right)\vec{\jmath} \qquad \text{on } U.$$

To find all scalar potentials $f \in C^1(U,\mathbb{R})$ satisfying $\nabla f = \vec{G}$ on U, solve

$$\frac{\partial f}{\partial x}(x,y) = \frac{2x}{y^2}, \qquad \frac{\partial f}{\partial y}(x,y) = \frac{1}{y} - \frac{2x^2}{y^3}.$$

Integrating the first equation with respect to x at fixed $y > 0$ gives

$$f(x,y) = \int \frac{2x}{y^2}\,dx = \frac{x^2}{y^2} + \theta(y), \qquad \theta \in C^1(]0,+\infty[,\mathbb{R}).$$

Differentiating with respect to y and matching the second equation yields

$$\frac{\partial f}{\partial y}(x,y) = -\frac{2x^2}{y^3} + \theta'(y) = \frac{1}{y} - \frac{2x^2}{y^3} \iff \theta'(y) = \frac{1}{y} \qquad \forall y > 0.$$

Integrating over the connected interval $]0, +\infty[$ (where $y > 0$) gives $\theta(y) = \ln y + C$ with $C \in \mathbb{R}$.

$$\boxed{\begin{aligned} \mu(y) &= \frac{1}{y^3} \quad (y > 0), \\ f(x,y) &= \frac{x^2}{y^2} + \ln y + C, \quad C \in \mathbb{R} \end{aligned}}$$

> ► **Common Pitfall:**
> On the connected upper half-plane $U = \{y > 0\}$, a single real constant $C \in \mathbb{R}$ parameterizes all primitives; had the domain been $\mathbb{R}^2 \setminus \{y = 0\}$, which has two connected components $\{y > 0\}$ and $\{y < 0\}$, two independent constants C_+ and C_- along with $\ln|y|$ would be required.

Exercise 4

1. Let Γ be the helical arc parameterized by $\gamma(t) = \big(2\cos t,\ 2\sin t,\ \sqrt{5}\,t\big)$ for $t \in [0, \pi]$.

(a) Compute the mass $M = \displaystyle\int_\Gamma \rho(x,y,z)\,ds$ of the wire Γ with linear density $\rho(x,y,z) = z + x^2 y$, and give the value of $\displaystyle\int_{\Gamma^-} \rho(x,y,z)\,ds$, where Γ^- denotes Γ with the opposite orientation. ★★☆☆☆

Method ***Scalar Line Integral Evaluation*** *($ds = \|\gamma'(t)\|\,dt$)* ***and Orientation Invariance***

> ***Why we do this (Way of Thinking):***
> A scalar line integral integrates ρ against the non-negative arc length element $ds = \|\gamma'(t)\|\,dt$. Along a circular helix, the horizontal velocity components combine via $\sin^2 t + \cos^2 t = 1$ to yield a constant speed $\|\gamma'(t)\|$. Furthermore, reversing the parameterization of Γ preserves the norm of the tangent vector up to parameter reflection, so scalar line integrals of the first kind are invariant under orientation reversal.

Execution: Differentiate $\gamma(t) = \big(2\cos t,\ 2\sin t,\ \sqrt{5}\,t\big)$ for $t \in [0, \pi]$:

$$\gamma'(t) = \big(-2\sin t,\ 2\cos t,\ \sqrt{5}\big).$$

The Euclidean norm of $\gamma'(t)$ is constant on $[0, \pi]$:

$$\|\gamma'(t)\| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (\sqrt{5})^2} = \sqrt{4(\sin^2 t + \cos^2 t) + 5} = \sqrt{4+5} = \sqrt{9} = 3,$$

so $ds = \|\gamma'(t)\|\,dt = 3\,dt$. Evaluating $\rho(x,y,z) = z + x^2 y$ along $\gamma(t)$ (where $\rho(\gamma(t)) \geq 0$ for all $t \in [0,\pi]$ since $t \geq 0$ and $\sin t \geq 0$) gives

$$\rho\big(\gamma(t)\big) = \sqrt{5}\,t + (2\cos t)^2(2\sin t) = \sqrt{5}\,t + 4\cos^2 t \cdot (2\sin t) = \sqrt{5}\,t + 8\cos^2 t \sin t.$$

Substituting $\rho(\gamma(t))$ and $ds = 3\,dt$ into M yields

$$\begin{aligned} M = \int_\Gamma \rho(x,y,z)\,ds &= \int_0^\pi \big(\sqrt{5}\,t + 8\cos^2 t \sin t\big) \cdot 3\,dt \\ &= 3\sqrt{5}\int_0^\pi t\,dt + 24\int_0^\pi \cos^2 t \sin t\,dt. \end{aligned}$$

The first integral equals

$$3\sqrt{5}\int_0^{\pi} t\,dt = 3\sqrt{5}\left[\frac{t^2}{2}\right]_0^{\pi} = \frac{3\sqrt{5}}{2}\pi^2.$$

For the trigonometric integral:

Sub **Set $u = \cos t$, so $du = -\sin t\,dt$; the bounds map $t = 0 \mapsto u = 1$ and $t = \pi \mapsto u = -1$.**

$$24\int_0^{\pi} \cos^2 t \sin t\,dt = 24\int_1^{-1} u^2(-du) = 24\int_{-1}^{1} u^2\,du = 24\left[\frac{u^3}{3}\right]_{-1}^{1} = 24\cdot\frac{1-(-1)}{3} = 16.$$

Hence $M = \dfrac{3\sqrt{5}}{2}\pi^2 + 16$.

To evaluate $\displaystyle\int_{\Gamma^-} \rho(x,y,z)\,ds$, parameterize Γ^- by $\gamma^-(s) = \gamma(\pi - s)$ for $s \in [0,\pi]$. By the chain rule, $(\gamma^-)'(s) = -\gamma'(\pi - s)$, whence $\|(\gamma^-)'(s)\| = \|\gamma'(\pi - s)\|$. Applying the change of variable $t = \pi - s$ (***dt = −ds***, mapping $s = 0 \mapsto t = \pi$ and $s = \pi \mapsto t = 0$) gives

$$\begin{aligned}\int_{\Gamma^-} \rho(x,y,z)\,ds &= \int_0^{\pi} \rho\big(\gamma(\pi - s)\big)\,\|\gamma'(\pi - s)\|\,ds \\ &= \int_{\pi}^{0} \rho\big(\gamma(t)\big)\,\|\gamma'(t)\|\,(-dt) = \int_0^{\pi} \rho\big(\gamma(t)\big)\,\|\gamma'(t)\|\,dt = M.\end{aligned}$$

$$\boxed{M = \int_{\Gamma} \rho(x,y,z)\,ds = \int_{\Gamma^-} \rho(x,y,z)\,ds = \frac{3\sqrt{5}}{2}\pi^2 + 16}$$

> ▸ **Common Pitfall:**
> Two errors arise regularly in this calculation: writing $(2\cos t)^2(2\sin t) = 4\cos^2 t \sin t$ instead of $8\cos^2 t \sin t$, and negating the integral over Γ^- by confusing scalar line integrals $\int_\Gamma \rho\,ds$ (which are independent of orientation) with vector line integrals $\int_\Gamma \vec{F}\cdot d\vec{s}$ (which change sign under orientation reversal).

(b) Prove that $\dfrac{3\pi}{\sqrt{9+25\pi^4}} \le \displaystyle\int_\Gamma \frac{ds}{\sqrt{5+x^2+y^2+z^4}} \le \pi$**.** ★★☆☆☆

Method ***Pointwise Bounding on Γ and Comparison Inequality for Scalar Line Integrals***

> ***Why we do this (Way of Thinking):***
> The integral $\int_0^{\pi} 3\,(9+25t^4)^{-1/2}\,dt$ is an elliptic integral with no elementary antiderivative. Instead, because Γ lies on the circular cylinder $x^2 + y^2 = 4$ while $z = \sqrt{5}\,t$ varies over $[0, \sqrt{5}\,\pi]$, we obtain sharp upper and lower bounds for the integrand along Γ and multiply both bounds by the arc length $\ell(\Gamma) = 3\pi$.

Execution: For every point $(x,y,z) = \gamma(t) = \big(2\cos t,\ 2\sin t,\ \sqrt{5}\,t\big)$ with $t \in [0,\pi]$, we have

$$x(t)^2 + y(t)^2 = 4\cos^2 t + 4\sin^2 t = 4(\cos^2 t + \sin^2 t) = 4,$$

and $0 \le t \le \pi$, which implies

$$0 \le z(t)^4 = \big(\sqrt{5}\,t\big)^4 = 25\,t^4 \le 25\,\pi^4.$$

Adding $5 + x(t)^2 + y(t)^2 = 5 + 4 = 9$ throughout gives

$$9 \le 5 + x^2 + y^2 + z^4 \le 9 + 25\pi^4 \qquad \forall (x,y,z) \in \Gamma.$$

Since $u \mapsto \dfrac{1}{\sqrt{u}}$ is strictly decreasing on $]0, +\infty[$, we obtain

$$\frac{1}{\sqrt{9+25\pi^4}} \le \frac{1}{\sqrt{5+x^2+y^2+z^4}} \le \frac{1}{\sqrt{9}} = \frac{1}{3} \qquad \forall (x,y,z) \in \Gamma.$$

From part (a), $\|\gamma'(t)\| = 3$ on $[0,\pi]$, so the arc length of Γ is

$$\ell(\Gamma) = \int_\Gamma ds = \int_0^\pi \|\gamma'(t)\|\,dt = \int_0^\pi 3\,dt = 3\pi.$$

By the bounding property of scalar line integrals ($\alpha\,\ell(\Gamma) \le \int_\Gamma f\,ds \le \beta\,\ell(\Gamma)$ whenever $\alpha \le f \le \beta$ on Γ), multiplying the pointwise inequality by $\ell(\Gamma) = 3\pi > 0$ yields

$$\frac{3\pi}{\sqrt{9+25\pi^4}} \le \int_\Gamma \frac{ds}{\sqrt{5+x^2+y^2+z^4}} \le \frac{1}{3}\cdot 3\pi = \pi.$$

$$\boxed{\frac{3\pi}{\sqrt{9+25\pi^4}} \le \int_\Gamma \frac{ds}{\sqrt{5+x^2+y^2+z^4}} \le \pi} \quad \blacksquare$$

2. Fix $a > 0$ and let C be the cardioid parameterized by $\gamma(t) = \big(a(1+\cos t)\cos t,\ a(1+\cos t)\sin t\big)$ for $t \in [0,2\pi]$. Compute $\oint_C |y|\,ds$. ★★★☆☆

Method *Polar Speed Reduction, Half-Angle Identities, and Interval Splitting at $t = \pi$*

> ***Why we do this (Way of Thinking):***
> Viewing $\gamma(t) = \big(r(t)\cos t,\ r(t)\sin t\big)$ in polar form with $r(t) = a(1+\cos t)$ reduces $x'(t)^2 + y'(t)^2$ to $r(t)^2 + r'(t)^2 = 2a^2(1+\cos t) = 4a^2\cos^2(t/2)$. Taking the square root requires retaining the absolute value $\|\gamma'(t)\| = 2a\,|\cos(t/2)|$. Because both $\sin t$ (inside $|y|$) and $\cos(t/2)$ (inside $\|\gamma'(t)\|$) change sign at $t = \pi$, splitting $[0,2\pi]$ into $[0,\pi] \cup [\pi,2\pi]$ and reflecting the second half onto $[0,\pi]$ removes all absolute values cleanly.

Execution: Set $r(t) = a(1+\cos t)$ on $[0,2\pi]$, so $r'(t) = -a\sin t$, $x(t) = r(t)\cos t$, and $y(t) = r(t)\sin t$. Differentiating with respect to t gives

$$x'(t) = r'(t)\cos t - r(t)\sin t = -a\sin t\cos t - a(1+\cos t)\sin t = -a\sin t\,(1+2\cos t),$$
$$y'(t) = r'(t)\sin t + r(t)\cos t = -a\sin^2 t + a(1+\cos t)\cos t = a(\cos t + \cos^2 t - \sin^2 t).$$

Squaring and adding the two derivative expressions, where the cross terms $\pm 2r(t)r'(t)\sin t\cos t$ cancel, yields

$$\begin{aligned}\|\gamma'(t)\|^2 &= \big(r'(t)\cos t - r(t)\sin t\big)^2 + \big(r'(t)\sin t + r(t)\cos t\big)^2\\ &= r'(t)^2(\cos^2 t + \sin^2 t) + r(t)^2(\sin^2 t + \cos^2 t)\\ &= (-a\sin t)^2 + a^2(1+\cos t)^2\\ &= a^2\big(\sin^2 t + 1 + 2\cos t + \cos^2 t\big) = 2a^2(1+\cos t).\end{aligned}$$

Applying the half-angle identity $1 + \cos t = 2\cos^2\left(\frac{t}{2}\right)$ together with $a > 0$ gives

$$\|\gamma'(t)\| = \sqrt{4a^2\cos^2\left(\frac{t}{2}\right)} = 2a\left|\cos\left(\frac{t}{2}\right)\right|,$$

which vanishes only at the isolated cusp $t = \pi$. Since $a > 0$ and $1 + \cos t \ge 0$, we also have $|y(t)| = a(1+\cos t)\,|\sin t|$. Split $[0,2\pi]$ at $t = \pi$:

$$\oint_C |y|\,ds = \int_0^{2\pi} 2a^2(1+\cos t)\,|\sin t|\left|\cos\left(\frac{t}{2}\right)\right| dt = I_1 + I_2,$$

where I_1 integrates over $[0,\pi]$ and I_2 integrates over $[\pi,2\pi]$.

Sub **In I_2, set $s = 2\pi - t$, so $ds = -dt$; the bounds map $t = \pi \mapsto s = \pi$ and $t = 2\pi \mapsto s = 0$.**

Using $\cos(2\pi - s) = \cos s$, $|\sin(2\pi - s)| = |-\sin s| = |\sin s|$, and $\left|\cos\left(\pi - \frac{s}{2}\right)\right| = \left|-\cos\left(\frac{s}{2}\right)\right| = \left|\cos\left(\frac{s}{2}\right)\right|$, we obtain

$$I_2 = \int_{\pi}^{0} 2a^2(1+\cos s)\,|\sin s|\,\left|\cos\left(\frac{s}{2}\right)\right|(-ds) = \int_{0}^{\pi} 2a^2(1+\cos s)\,|\sin s|\,\left|\cos\left(\frac{s}{2}\right)\right| ds = I_1.$$

On $[0,\pi]$, we have $\frac{t}{2} \in \left[0, \frac{\pi}{2}\right]$, so $\cos\left(\frac{t}{2}\right) \geq 0$, $\sin\left(\frac{t}{2}\right) \geq 0$, and $\sin t = 2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right) \geq 0$. Substituting $1+\cos t = 2\cos^2\left(\frac{t}{2}\right)$ and $\sin t = 2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)$ into $2I_1$ gives

$$\begin{aligned}\oint_C |y|\,ds = 2I_1 &= 4a^2 \int_0^{\pi} \left(2\cos^2\frac{t}{2}\right)\left(2\sin\frac{t}{2}\cos\frac{t}{2}\right)\cos\frac{t}{2}\,dt \\ &= 16a^2 \int_0^{\pi} \cos^4\left(\frac{t}{2}\right)\sin\left(\frac{t}{2}\right)dt.\end{aligned}$$

Sub **Set $u = \cos\left(\frac{t}{2}\right)$, so $du = -\frac{1}{2}\sin\left(\frac{t}{2}\right)dt$; the bounds map $t = 0 \mapsto u = 1$ and $t = \pi \mapsto u = 0$.**

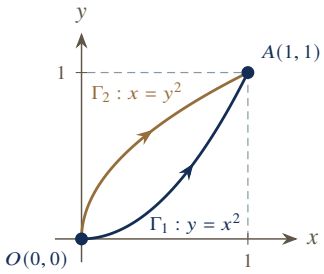
$$\oint_C |y|\,ds = 16a^2 \int_1^0 u^4(-2\,du) = 32a^2 \int_0^1 u^4\,du = 32a^2\left[\frac{u^5}{5}\right]_0^1 = \frac{32a^2}{5}.$$

$$\boxed{\oint_C |y|\,ds = \frac{32a^2}{5}}$$

> ▸ **Common Pitfall:**
> Writing $\sqrt{\cos^2(t/2)} = \cos(t/2)$ without absolute values across $[0, 2\pi]$ assigns a negative sign to $\|\gamma'(t)\|$ on $]\pi, 2\pi]$, causing the contributions of $[0,\pi]$ and $[\pi, 2\pi]$ to cancel to 0.

> ✓ **Sanity Check:**
> Since y and ds each have dimension $[a]$, the integral scales as a^2. Moreover, $\max_C |y| = \frac{3\sqrt{3}}{4}a \approx 1.30\,a$ and the perimeter of the cardioid is $\ell(C) = \int_0^{2\pi} 2a|\cos(t/2)|\,dt = 8a$, so $0 < \oint_C |y|\,ds < 10.4\,a^2$, consistent with $\frac{32}{5}a^2 = 6.4\,a^2$.

3. For $\lambda \in \mathbb{R}$, let $\vec{F}_\lambda(x,y) = (2xy + y^2)\,\vec{i} + (\lambda x^2 + 2xy)\,\vec{j}$ on $\mathbb{R}^2$. Consider the two oriented parabolic arcs $\Gamma_1 : y = x^2$ and $\Gamma_2 : x = y^2$ from $O(0,0)$ to $A(1,1)$ shown below. ★★☆☆☆



Compute $W_1(\lambda) = \int_{\Gamma_1} \vec{F}_\lambda \cdot d\vec{s}$ and $W_2(\lambda) = \int_{\Gamma_2} \vec{F}_\lambda \cdot d\vec{s}$. Determine the value of λ for which $W_1(\lambda) = W_2(\lambda)$, show that $\vec{F}_\lambda$ is then conservative on $\mathbb{R}^2$, and deduce $\int_\Gamma \vec{F}_\lambda \cdot d\vec{s}$ for every piecewise C^1 curve Γ from O to

A.

Method ***Parameterized Work Integrals, Poincaré's Theorem on $\mathbb{R}^2$, and Fundamental Theorem for Line Integrals***

> ***Why we do this (Way of Thinking):***
> Pulling back $\vec{F}_\lambda \cdot d\vec{s}$ along $\gamma_1(t) = (t, t^2)$ and $\gamma_2(t) = (t^2, t)$ yields two affine expressions in λ whose equality isolates a unique candidate λ. Because equal circulation along two particular arcs does not guarantee path independence across all curves, we verify $\partial_y P_\lambda = \partial_x Q_\lambda$ on star-shaped $\mathbb{R}^2$, integrate a scalar potential f, and evaluate $f(A) - f(O)$ for an arbitrary piecewise C^1 path from O to A.

Execution:

- **Work along $\Gamma_1 : y = x^2$:** Parameterize Γ_1 from $O(0,0)$ to $A(1,1)$ by $\gamma_1(t) = (t, t^2)$ for $t \in [0,1]$, with $\gamma_1'(t) = (1, 2t)$. Then

$$\begin{aligned}\vec{F}_\lambda\big(\gamma_1(t)\big) \cdot \gamma_1'(t) &= \big(2t \cdot t^2 + (t^2)^2\big) \cdot 1 + \big(\lambda t^2 + 2t \cdot t^2\big) \cdot 2t \\ &= (2t^3 + t^4) + (2\lambda t^3 + 4t^4) = 2(\lambda + 1)t^3 + 5t^4.\end{aligned}$$

  Integrating over $[0,1]$ gives

$$W_1(\lambda) = \int_0^1 \big(2(\lambda+1)t^3 + 5t^4\big)\, dt = \left[\frac{\lambda+1}{2}\, t^4 + t^5\right]_0^1 = \frac{\lambda+1}{2} + 1 = \frac{\lambda+3}{2}.$$

- **Work along $\Gamma_2 : x = y^2$:** Parameterize Γ_2 from $O(0,0)$ to $A(1,1)$ by $\gamma_2(t) = (t^2, t)$ for $t \in [0,1]$, with $\gamma_2'(t) = (2t, 1)$. Then

$$\begin{aligned}\vec{F}_\lambda\big(\gamma_2(t)\big) \cdot \gamma_2'(t) &= \big(2t^2 \cdot t + t^2\big) \cdot 2t + \big(\lambda(t^2)^2 + 2t^2 \cdot t\big) \cdot 1 \\ &= (4t^4 + 2t^3) + (\lambda t^4 + 2t^3) = 4t^3 + (\lambda + 4)t^4.\end{aligned}$$

  Integrating over $[0,1]$ gives

$$W_2(\lambda) = \int_0^1 \big(4t^3 + (\lambda+4)t^4\big)\, dt = \left[t^4 + \frac{\lambda+4}{5}\, t^5\right]_0^1 = 1 + \frac{\lambda+4}{5} = \frac{\lambda+9}{5}.$$

Equating the two path integrals yields

$$W_1(\lambda) = W_2(\lambda) \iff \frac{\lambda+3}{2} = \frac{\lambda+9}{5} \iff 5(\lambda+3) = 2(\lambda+9) \iff 3\lambda = 3 \iff \lambda = 1.$$

For $\lambda = 1$, write $\vec{F}_1(x,y) = P_1(x,y)\,\vec{i} + Q_1(x,y)\,\vec{j}$ with $P_1(x,y) = 2xy + y^2$ and $Q_1(x,y) = x^2 + 2xy$. Both components belong to $C^\infty(\mathbb{R}^2, \mathbb{R})$, and

$$\frac{\partial P_1}{\partial y}(x,y) = \frac{\partial}{\partial y}(2xy + y^2) = 2x + 2y, \qquad \frac{\partial Q_1}{\partial x}(x,y) = \frac{\partial}{\partial x}(x^2 + 2xy) = 2x + 2y.$$

Since $\dfrac{\partial P_1}{\partial y} = \dfrac{\partial Q_1}{\partial x}$ everywhere on the convex (hence star-shaped) plane $\mathbb{R}^2$, Poincaré's theorem guarantees that $\vec{F}_1$ is conservative on $\mathbb{R}^2$. To recover a scalar potential $f \in C^1(\mathbb{R}^2, \mathbb{R})$ satisfying $\nabla f = \vec{F}_1$, integrate $\partial_x f = 2xy + y^2$ with respect to x:

$$f(x,y) = \int (2xy + y^2)\, dx = x^2 y + xy^2 + \varphi(y), \qquad \varphi \in C^1(\mathbb{R}, \mathbb{R}).$$

Differentiating with respect to y gives $\dfrac{\partial f}{\partial y}(x,y) = x^2 + 2xy + \varphi'(y) = Q_1(x,y) + \varphi'(y)$, so $\varphi'(y) = 0$ and $\varphi(y) = C \in \mathbb{R}$. Choosing $C = 0$ gives $f(x,y) = x^2 y + xy^2$.

By the Fundamental Theorem for Line Integrals, for every piecewise C^1 curve Γ in $\mathbb{R}^2$ oriented from $O(0,0)$ to $A(1,1)$,

$$\int_\Gamma \vec{F}_1 \cdot d\vec{s} = f(1,1) - f(0,0) = (1^2 \cdot 1 + 1 \cdot 1^2) - (0+0) = 2.$$

$$W_1(\lambda) = \frac{\lambda+3}{2}, \qquad W_2(\lambda) = \frac{\lambda+9}{5},$$
$$W_1(\lambda) = W_2(\lambda) \iff \lambda = 1, \qquad \int_\Gamma \vec{F}_1 \cdot d\vec{s} = 2$$

▸ **Common Pitfall:**
Equality of the line integrals along two specific curves Γ_1 and Γ_2 does not by itself prove that a vector field is conservative; conservativeness on $\mathbb{R}^2$ must be established via Poincaré's theorem ($\partial_y P_1 = \partial_x Q_1$ on star-shaped $\mathbb{R}^2$) or by exhibiting a global potential f satisfying $\nabla f = \vec{F}_1$.

4. Let Γ be the oriented planar curve parameterized by $\gamma(t) = \big(\cos t + \sin(2t),\ \sin t + \sin^2 t\big)$ for $t \in [0, \pi]$. Compute ★★★☆☆

$$I = \int_\Gamma \frac{1 - x^2 - 2xy + y^2}{(1 + x^2 + y^2)^2}\, dx + \frac{1 + x^2 - 2xy - y^2}{(1 + x^2 + y^2)^2}\, dy.$$

Method ***Quotient-Rule Primitive Construction on $\mathbb{R}^2$ and Endpoint Evaluation***

Why we do this (Way of Thinking):
Pulling back $(1 + x^2 + y^2)^{-2}$ along $\gamma(t) = \big(\cos t + \sin(2t),\ \sin t + \sin^2 t\big)$ produces a high-degree trigonometric rational integrand, which indicates that we should first check whether $\omega = P\,dx + Q\,dy$ is exact on $\mathbb{R}^2$. Two structural features guide the integration: first, $1 + x^2 + y^2 \geq 1 > 0$ has no singularities anywhere on $\mathbb{R}^2$; second, the denominator $(1 + x^2 + y^2)^2$ matches the quotient-rule square v^2 for $v(x, y) = 1 + x^2 + y^2$. Decomposing the numerator of P as $1 \cdot (1 + x^2 + y^2) - (x + y)(2x)$ identifies the global primitive $f(x, y) = \frac{x+y}{1+x^2+y^2}$ and reduces I to $f(\gamma(\pi)) - f(\gamma(0))$.

Execution: Write $\omega = P(x, y)\,dx + Q(x, y)\,dy$ with

$$P(x, y) = \frac{1 - x^2 - 2xy + y^2}{(1 + x^2 + y^2)^2}, \qquad Q(x, y) = \frac{1 + x^2 - 2xy - y^2}{(1 + x^2 + y^2)^2}.$$

For all $(x, y) \in \mathbb{R}^2$, we have $1 + x^2 + y^2 \geq 1 > 0$, so $P, Q \in C^\infty(\mathbb{R}^2, \mathbb{R})$.

To integrate $\frac{\partial f}{\partial x}(x, y) = P(x, y)$, rewrite the numerator of P in terms of $v(x, y) = 1 + x^2 + y^2$ and $\frac{\partial v}{\partial x}(x, y) = 2x$:

$$1 - x^2 - 2xy + y^2 = (1 + x^2 + y^2) - (2x^2 + 2xy) = 1 \cdot (1 + x^2 + y^2) - (x + y) \cdot (2x).$$

By the quotient rule with respect to x at fixed $y \in \mathbb{R}$,

$$P(x, y) = \frac{\frac{\partial}{\partial x}(x + y) \cdot (1 + x^2 + y^2) - (x + y) \cdot \frac{\partial}{\partial x}(1 + x^2 + y^2)}{(1 + x^2 + y^2)^2} = \frac{\partial}{\partial x}\left(\frac{x + y}{1 + x^2 + y^2}\right).$$

Integrating with respect to x gives candidates of the form $f(x, y) = \frac{x + y}{1 + x^2 + y^2} + \varphi(y)$. Differentiating $\frac{x + y}{1 + x^2 + y^2}$ with respect to y via the quotient rule yields

$$\begin{aligned}
\frac{\partial}{\partial y}\left(\frac{x + y}{1 + x^2 + y^2}\right) &= \frac{\frac{\partial}{\partial y}(x + y) \cdot (1 + x^2 + y^2) - (x + y) \cdot \frac{\partial}{\partial y}(1 + x^2 + y^2)}{(1 + x^2 + y^2)^2} \\
&= \frac{1 \cdot (1 + x^2 + y^2) - (x + y)(2y)}{(1 + x^2 + y^2)^2} \\
&= \frac{1 + x^2 + y^2 - 2xy - 2y^2}{(1 + x^2 + y^2)^2} = \frac{1 + x^2 - 2xy - y^2}{(1 + x^2 + y^2)^2} = Q(x, y).
\end{aligned}$$

Hence $\varphi'(y) = 0$, and setting $\varphi(y) = 0$ defines the smooth global primitive

$$f(x,y) = \frac{x+y}{1+x^2+y^2}, \qquad df = P\,dx + Q\,dy = \omega \quad \text{on } \mathbb{R}^2.$$

(In particular, by Schwarz's theorem, $\partial_y P = \partial^2_{yx} f = \partial^2_{xy} f = \partial_x Q$ holds identically on $\mathbb{R}^2$.)

Since $\gamma : [0,\pi] \to \mathbb{R}^2$ is of class C^1 on $[0,\pi]$, compute its endpoints:

$$\gamma(0) = \big(\cos 0 + \sin 0,\ \sin 0 + \sin^2 0\big) = (1+0,\ 0+0) = (1,0),$$
$$\gamma(\pi) = \big(\cos\pi + \sin(2\pi),\ \sin\pi + \sin^2\pi\big) = (-1+0,\ 0+0) = (-1,0).$$

Applying the Fundamental Theorem for Line Integrals gives

$$I = \int_\Gamma \omega = \int_\Gamma df = f\big(\gamma(\pi)\big) - f\big(\gamma(0)\big) = f(-1,0) - f(1,0) = \frac{-1+0}{1+(-1)^2+0^2} - \frac{1+0}{1+1^2+0^2} = -\frac{1}{2} - \frac{1}{2} = -1.$$

$$\boxed{I = f(-1,0) - f(1,0) = -1}$$

> ▸ **Common Pitfall:**
> Before replacing $\int_\Gamma \omega$ by $f(\gamma(\pi)) - f(\gamma(0))$, confirm that ω has no singularities anywhere on a domain containing Γ. Here $1 + x^2 + y^2 \geq 1 > 0$ guarantees smoothness across all of $\mathbb{R}^2$, whereas denominators such as $x^2 + y^2$ fail at the origin and can invalidate path independence.

> ✓ **Sanity Check:**
> Because $\omega = df$ is exact on $\mathbb{R}^2$, replacing Γ with the horizontal segment from $(1,0)$ to $(-1,0)$ along $y = 0$ ($dy = 0$) gives $\int_1^{-1} \frac{1-x^2}{(1+x^2)^2}\,dx = \left[\frac{x}{1+x^2}\right]_1^{-1} = -\frac{1}{2} - \frac{1}{2} = -1$, matching $f(-1,0) - f(1,0)$.

Exercise 5

On the punctured plane $\Omega = \mathbb{R}^2 \setminus \{(0,0)\}$, consider the differential 1-form

$$\omega = \left(2xy - \frac{2y}{x^2+4y^2}\right)dx + \left(x^2 + \frac{2x}{x^2+4y^2}\right)dy.$$

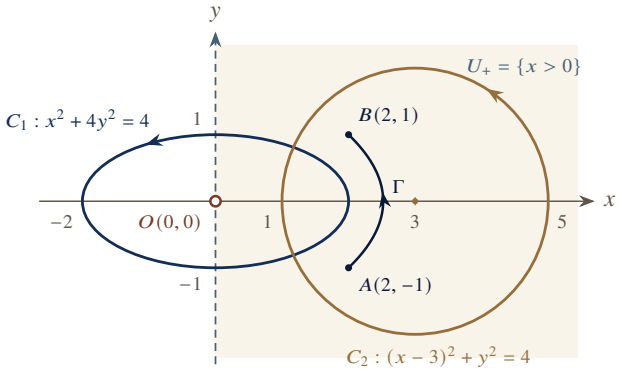


Figure 1: Geometry on $\Omega = \mathbb{R}^2 \setminus \{(0,0)\}$: the ellipse C_1 winds around the puncture $O(0,0)$, whereas the circle C_2 and the path Γ from $A(2,-1)$ to $B(2,1)$ lie inside the convex right half-plane $U_+ = \{x > 0\}$.

1. Show that ω is closed on Ω. Why does Poincaré's theorem not apply to ω on Ω? ★★☆☆☆

Method *Cross-Derivative Closedness Test ($\partial_y P = \partial_x Q$) and Topological Check of Poincaré's Domain*

Hypothesis

> ***Why we do this (Way of Thinking):***
> Write $\omega = P(x,y)\,dx + Q(x,y)\,dy$. By definition, a C^1 differential 1-form on an open set $\Omega \subset \mathbb{R}^2$ is closed when $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ everywhere on Ω. Here x^2+4y^2 vanishes solely at $(0,0) \notin \Omega$, so P and Q are smooth on Ω and their cross partial derivatives follow from the quotient rule. Once closedness is established, recall the domain hypothesis of Poincaré's theorem: closedness implies exactness automatically only when the underlying open set is star-shaped or simply connected. Removing $(0,0)$ creates a puncture that prevents segments through the origin and loops around the origin from staying inside Ω.

Execution: Step 1 — Regularity and closedness on Ω: Set

$$P(x,y) = 2xy - \frac{2y}{x^2+4y^2}, \qquad Q(x,y) = x^2 + \frac{2x}{x^2+4y^2}.$$

For every $(x,y) \in \Omega = \mathbb{R}^2 \setminus \{(0,0)\}$, at least one of x or y is nonzero, so $x^2+4y^2 > 0$. As rational functions with a strictly positive denominator on the open set Ω, P and Q belong to $C^1(\Omega, \mathbb{R})$.

Differentiating P with respect to y and Q with respect to x via the quotient rule:

$$\begin{aligned}
\frac{\partial P}{\partial y}(x,y) &= \frac{\partial}{\partial y}(2xy) - \frac{\partial}{\partial y}\left(\frac{2y}{x^2+4y^2}\right) \\
&= 2x - \frac{2(x^2+4y^2) - 2y(8y)}{(x^2+4y^2)^2} \\
&= 2x - \frac{2x^2+8y^2-16y^2}{(x^2+4y^2)^2} = 2x - \frac{2x^2-8y^2}{(x^2+4y^2)^2} = 2x + \frac{8y^2-2x^2}{(x^2+4y^2)^2},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial Q}{\partial x}(x,y) &= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}\left(\frac{2x}{x^2+4y^2}\right) \\
&= 2x + \frac{2(x^2+4y^2) - 2x(2x)}{(x^2+4y^2)^2} \\
&= 2x + \frac{2x^2+8y^2-4x^2}{(x^2+4y^2)^2} = 2x + \frac{8y^2-2x^2}{(x^2+4y^2)^2}.
\end{aligned}$$

Hence $\frac{\partial P}{\partial y}(x,y) = \frac{\partial Q}{\partial x}(x,y)$ for all $(x,y) \in \Omega$, so ω is closed on Ω.

Step 2 — Failure of the domain hypothesis in Poincaré's theorem: Poincaré's theorem asserts that a closed C^1 differential 1-form on an open set U is exact provided U is star-shaped or simply connected. Neither property holds for $\Omega = \mathbb{R}^2 \setminus \{(0,0)\}$:

- **Not star-shaped:** Let $M_0 = (x_0, y_0) \in \Omega$ be any candidate star center. Since $(x_0,y_0) \neq (0,0)$, the opposite point $M_1 = (-x_0, -y_0)$ also belongs to Ω, yet the midpoint of the segment $[M_0, M_1]$ is $\frac{1}{2}M_0 + \frac{1}{2}M_1 = (0,0) \notin \Omega$, so $[M_0, M_1] \not\subset \Omega$.
- **Not simply connected:** Closed curves in Ω that encircle the origin (such as C_1 in Figure 1) enclose the missing point $(0,0) \notin \Omega$ and cannot be continuously shrunk to a point within Ω.

Therefore, Poincaré's theorem does not apply to ω on Ω.

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 2x + \frac{8y^2-2x^2}{(x^2+4y^2)^2} \ (\omega \text{ closed on } \Omega); \quad \Omega \text{ is neither star-shaped nor simply connected}$$

> ► **Common Pitfall:**
> Showing that Poincaré's theorem does not apply does *not* prove by itself that ω fails to be exact on Ω. Simple connectedness of Ω is a *sufficient* condition for closed forms to be exact, not a necessary condition for an individual form: for instance, $\tilde{\omega} = \frac{2x\,dx + 8y\,dy}{x^2+4y^2} = d\big(\ln(x^2+4y^2)\big)$ is exact on Ω despite the hole at $(0,0)$. Proving non-exactness requires finding a closed loop in Ω with nonzero circulation (Question 2).

> ✓ **Sanity Check:**
> Evaluate both cross partial derivatives at $(1,1) \in \Omega$, where $x^2+4y^2=5$: both $\frac{\partial P}{\partial y}(1,1)$ and $\frac{\partial Q}{\partial x}(1,1)$ equal $2(1)+\frac{8(1)-2(1)}{25} = 2+\frac{6}{25} = \frac{56}{25}$, confirming all quotient-rule signs.

2. Let C_1 be the ellipse $x^2+4y^2=4$ oriented counterclockwise. Compute $\oint_{C_1}\omega$. Is ω exact on Ω? Justify.

★★★☆☆

Method ***Exact-Differential Decomposition, Trigonometric Ellipse Parameterization, and Closed-Loop Exactness Criterion***

> ***Why we do this (Way of Thinking):***
> Two structural features simplify $\oint_{C_1}\omega$ immediately. First, the polynomial part $2xy\,dx + x^2\,dy$ equals the exact differential $d(x^2y)$ on all of $\mathbb{R}^2$, so its circulation around the closed curve C_1 vanishes without any trigonometric integration. Second, the denominator x^2+4y^2 of the remaining rational 1-form matches the defining equation $x^2+4y^2=4$ of C_1: parameterizing $\left(\frac{x}{2}\right)^2+y^2=1$ by $x=2\cos t$, $y=\sin t$ makes the denominator equal to the constant 4 and reduces the numerator to $\cos^2 t+\sin^2 t=1$. Finally, if ω were exact on Ω, its integral over *every* closed curve in Ω would vanish.

Execution: Step 1 — Separating the globally exact polynomial part: On $\mathbb{R}^2$, observe that

$$d(x^2y) = \frac{\partial(x^2y)}{\partial x}\,dx + \frac{\partial(x^2y)}{\partial y}\,dy = 2xy\,dx + x^2\,dy.$$

Hence on Ω we decompose $\omega = d(x^2y) + \omega_1$, where

$$\omega_1 = -\frac{2y}{x^2+4y^2}\,dx + \frac{2x}{x^2+4y^2}\,dy.$$

Since C_1 is a closed C^1 curve in Ω ($x^2+4y^2=4>0$ on C_1) and $d(x^2y)$ is exact on $\mathbb{R}^2$, the line integral of $d(x^2y)$ around C_1 vanishes:

$$\oint_{C_1} d(x^2y) = 0 \quad\Longrightarrow\quad \oint_{C_1}\omega = \oint_{C_1} d(x^2y) + \oint_{C_1}\omega_1 = \oint_{C_1}\omega_1.$$

Step 2 — Counterclockwise parameterization of C_1 and line integral: Divide $x^2+4y^2=4$ by 4 to obtain the standard ellipse form $\left(\frac{x}{2}\right)^2+y^2=1$ with semi-axes $a=2$ and $b=1$.

Sub **Set $x(t)=2\cos t$ and $y(t)=\sin t$ for $t\in[0,2\pi]$ (counterclockwise), so $dx=-2\sin t\,dt$ and $dy=\cos t\,dt$.**

Along C_1, $x(t)^2+4y(t)^2 = 4\cos^2 t + 4\sin^2 t = 4$. Substituting into $\oint_{C_1}\omega_1$:

$$\begin{aligned}\oint_{C_1}\omega &= \int_0^{2\pi}\left(-\frac{2\sin t}{4}(-2\sin t) + \frac{2(2\cos t)}{4}(\cos t)\right)dt \\ &= \int_0^{2\pi}\left(\frac{4\sin^2 t}{4}+\frac{4\cos^2 t}{4}\right)dt \\ &= \int_0^{2\pi}(\sin^2 t+\cos^2 t)\,dt = \int_0^{2\pi} 1\,dt = \Big[t\Big]_{t=0}^{t=2\pi} = 2\pi.\end{aligned}$$

Step 3 — Conclusion on exactness over Ω: Recall from the course that if a differential 1-form is exact on an open set Ω, then its line integral over every closed C^1 curve contained in Ω equals 0. Since $C_1 \subset \Omega$ is a closed

C^1 curve satisfying $\oint_{C_1} \omega = 2\pi \neq 0$, ω is **not exact** on Ω.

$$\oint_{C_1} \omega = 2\pi \neq 0 \quad \Longrightarrow \quad \omega \text{ is not exact on } \Omega$$

► **Common Pitfall:**
Parameterizing $x^2 + 4y^2 = 4$ as $x = 2\cos t$, $y = 2\sin t$ traces the circle $x^2 + y^2 = 4$ rather than the ellipse $\frac{x^2}{4} + y^2 = 1$ (since $(2\cos t)^2 + 4(2\sin t)^2 = 4\cos^2 t + 16\sin^2 t \neq 4$). Always normalize the ellipse equation to $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ before reading $(a, b) = (2, 1)$.

✓ **Sanity Check:**
Direct evaluation without separating $d(x^2 y)$ confirms the cancellation: along $x = 2\cos t$, $y = \sin t$, the polynomial part gives $\int_0^{2\pi} \left(4\cos t \sin t(-2\sin t) + 4\cos^2 t(\cos t)\right) dt = \int_0^{2\pi} \left(-8\sin^2 t \cos t + 4(1 - \sin^2 t)\cos t\right) dt = \left[4\sin t - 4\sin^3 t\right]_0^{2\pi} = 0$, leaving 2π.

3. Let C_2 be the circle $(x-3)^2 + y^2 = 4$ oriented counterclockwise. Compute $\oint_{C_2} \omega$. ★★☆☆☆

Method *Green–Riemann Formula on a Compact Disk Disjoint from the Singularity*

Why we do this (Way of Thinking):
Substituting $x = 3 + 2\cos t$, $y = 2\sin t$ directly into ω produces the denominator $(3 + 2\cos t)^2 + 16\sin^2 t$, which is cumbersome to integrate in t. Instead, inspect the closed disk $D_2 = \{(x-3)^2 + y^2 \leq 4\}$ bounded by C_2 (Figure 1): because its center is $(3, 0)$ and its radius is 2, every point in D_2 satisfies $x \geq 3 - 2 = 1 > 0$. Thus $(0, 0) \notin D_2$, so P and Q are C^1 throughout an open neighborhood of D_2, and the Green–Riemann formula converts $\oint_{C_2} \omega$ into $\iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = 0$.

Execution: Let $D_2 = \left\{(x, y) \in \mathbb{R}^2 : (x-3)^2 + y^2 \leq 4\right\}$ denote the closed circular disk bounded by the counterclockwise circle $C_2 = \partial D_2^+$. For every $(x, y) \in D_2$,

$$(x-3)^2 \leq (x-3)^2 + y^2 \leq 4 \implies |x-3| \leq 2 \implies 1 \leq x \leq 5.$$

In particular, $x \geq 1 > 0$ on D_2, so D_2 is contained inside the open right half-plane $U_+ = \left\{(x, y) \in \mathbb{R}^2 : x > 0\right\} \subset \Omega$ (the singularity $(0, 0)$ lies strictly outside D_2). Since $P, Q \in C^1(U_+)$ and $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 0$ on U_+ by Question 1, applying the Green–Riemann formula to the elementary domain D_2 yields

$$\oint_{C_2} \omega = \oint_{\partial D_2^+} \left(P(x, y)\, dx + Q(x, y)\, dy\right) = \iint_{D_2} \left(\frac{\partial Q}{\partial x}(x, y) - \frac{\partial P}{\partial y}(x, y)\right) dx\, dy = \iint_{D_2} 0\, dx\, dy = 0.$$

$$\oint_{C_2} \omega = 0$$

► **Common Pitfall:**
Never invoke $\oint_{\partial D^+} \omega = \iint_D (\partial_x Q - \partial_y P)\, dx\, dy = 0$ without first proving $(0, 0) \notin D$. Writing the same line on the interior $D_1 = \{x^2 + 4y^2 \leq 4\}$ of C_1 in Question 2 would falsely yield 0 instead of 2π, because P and Q are discontinuous at $(0, 0) \in D_1$ and violate the C^1 hypothesis of Green–Riemann.

> ✓ **Sanity Check:**
> Since $C_2 \subset U_+ = \{x > 0\}$ and Question 4 proves independently that $\omega|_{U_+}$ is exact on U_+, the circulation of the exact form $\omega|_{U_+}$ around any closed curve $C_2 \subset U_+$ must vanish identically.

4. Show that the restriction of ω to the open right half-plane $U_+ = \{(x,y) \in \mathbb{R}^2 : x > 0\}$ is exact, find all its primitives on U_+, and compute $\int_\Gamma \omega$ along any C^1 curve $\Gamma \subset U_+$ joining $A(2,-1)$ to $B(2,1)$. ★★★☆☆

Method ***Poincaré's Theorem on a Convex Half-Plane, Partial Integration for Primitives, and Path Independence***

> ***Why we do this (Way of Thinking):***
> Restricting ω to the open right half-plane $U_+ =]0,+\infty[\times\mathbb{R}$ eliminates the hole at $(0,0)$: any line segment joining two points with positive x-coordinates has positive x-coordinates everywhere along it, so U_+ is convex (hence star-shaped), and Poincaré's theorem guarantees exactness of $\omega|_{U_+}$. To solve $\frac{\partial f}{\partial x} = P$ and $\frac{\partial f}{\partial y} = Q$ on U_+, integrate $Q(x,y) = x^2 + \frac{2x}{x^2+4y^2}$ with respect to y first: because $x > 0$, dividing numerator and denominator by x^2 rewrites $\frac{2x}{x^2+4y^2}$ as $\frac{2/x}{1+(2y/x)^2}$, which is the exact y-derivative of $\arctan\left(\frac{2y}{x}\right)$. Once a primitive f is obtained, $\int_\Gamma \omega = f(B) - f(A)$ holds for every C^1 curve $\Gamma \subset U_+$ from A to B.

Execution: Step 1 — Exactness of $\omega|_{U_+}$ via Poincaré's theorem: The set $U_+ =]0,+\infty[\times\mathbb{R}$ is an open subset of Ω. Moreover, U_+ is convex: for any $M_1 = (x_1,y_1) \in U_+$, $M_2 = (x_2,y_2) \in U_+$, and $t \in [0,1]$, we have $x_1 > 0$ and $x_2 > 0$, so

$$(1-t)x_1 + tx_2 > 0 \quad \Longrightarrow \quad (1-t)M_1 + tM_2 \in U_+.$$

Hence U_+ is star-shaped with respect to every point of U_+. Since $\omega|_{U_+}$ is of class C^1 and closed on U_+ by Question 1, Poincaré's theorem implies that $\omega|_{U_+}$ is exact on U_+.

Step 2 — Construction of all primitives on U_+: We seek all functions $f \in C^1(U_+,\mathbb{R})$ such that $df = \omega$ on U_+, i.e., for all $(x,y) \in U_+$,

$$\begin{cases} \dfrac{\partial f}{\partial x}(x,y) = 2xy - \dfrac{2y}{x^2+4y^2}, \\ \dfrac{\partial f}{\partial y}(x,y) = x^2 + \dfrac{2x}{x^2+4y^2}. \end{cases}$$

Since $x > 0$ on U_+, factor x^2 out of $x^2 + 4y^2$ in the second equation:

$$\frac{\partial f}{\partial y}(x,y) = x^2 + \frac{\frac{2}{x}}{1+\left(\frac{2y}{x}\right)^2}.$$

Sub **For fixed $x > 0$, let $u = \dfrac{2y}{x} \Longrightarrow du = \dfrac{2}{x}\,dy$, so** $\displaystyle\int \frac{2/x}{1+(2y/x)^2}\,dy = \int \frac{du}{1+u^2} = \arctan\left(\frac{2y}{x}\right)$.

Integrating with respect to $y \in \mathbb{R}$ at fixed $x > 0$ gives

$$f(x,y) = x^2 y + \arctan\left(\frac{2y}{x}\right) + g(x),$$

where $g \in C^1(]0,+\infty[,\mathbb{R})$ depends on x alone. Differentiating this expression partially with respect to $x > 0$:

$$\begin{aligned} \frac{\partial f}{\partial x}(x,y) &= 2xy + \frac{1}{1+\left(\frac{2y}{x}\right)^2}\cdot\left(-\frac{2y}{x^2}\right) + g'(x) \\ &= 2xy - \frac{2y}{x^2\left(1+\frac{4y^2}{x^2}\right)} + g'(x) = 2xy - \frac{2y}{x^2+4y^2} + g'(x). \end{aligned}$$

Equating with $\dfrac{\partial f}{\partial x}(x,y) = 2xy - \dfrac{2y}{x^2+4y^2}$ yields $g'(x) = 0$ for all $x \in]0,+\infty[$. Since the interval $]0,+\infty[$ is connected, $g(x) = C$ for an arbitrary constant $C \in \mathbb{R}$. Thus the primitives of ω on U_+ are

$$f(x,y) = x^2y + \arctan\left(\frac{2y}{x}\right) + C, \qquad C \in \mathbb{R}.$$

Step 3 — Path-independent evaluation along $\Gamma \subset U_+$: Choose the primitive $f_0(x,y) = x^2y + \arctan\left(\dfrac{2y}{x}\right)$ ($C = 0$). Because Γ is a C^1 curve contained entirely inside U_+ from $A(2,-1)$ to $B(2,1)$ and $\omega|_{U_+} = df_0$, the theorem on line integrals of exact differential forms gives

$$\begin{aligned}\int_\Gamma \omega &= f_0(2,1) - f_0(2,-1)\\ &= \left(2^2(1) + \arctan\left(\frac{2(1)}{2}\right)\right) - \left(2^2(-1) + \arctan\left(\frac{2(-1)}{2}\right)\right)\\ &= \big(4 + \arctan(1)\big) - \big(-4 + \arctan(-1)\big)\\ &= \left(4 + \frac{\pi}{4}\right) - \left(-4 - \frac{\pi}{4}\right) = 8 + \frac{\pi}{2}.\end{aligned}$$

$$f(x,y) = x^2y + \arctan\left(\frac{2y}{x}\right) + C \ \ (C \in \mathbb{R}) \text{ on } U_+; \qquad \int_\Gamma \omega = 8 + \frac{\pi}{2}$$

► **Common Pitfall:**
Two subtle domain traps must be respected: (1) $f(x,y) = x^2y + \arctan\left(\frac{2y}{x}\right)$ is *not* a primitive on all of Ω, because it is undefined along the line $x = 0$ and jumps by π across $x = 0$ for $y \neq 0$; and (2) the hypothesis $\Gamma \subset U_+$ is essential—a curve $\tilde{\Gamma} \subset \Omega$ from $A(2,-1)$ to $B(2,1)$ passing to the *left* of the origin would form with Γ^- a closed loop around $(0,0)$ and give $\int_{\tilde{\Gamma}} \omega = 8 + \frac{\pi}{2} - 2\pi = 8 - \frac{3\pi}{2}$.

✓ **Sanity Check:**
Verify directly along the vertical segment $\Gamma_0 : x = 2,\ y = t \in [-1,1]$ inside U_+ ($dx = 0$, $dy = dt$): $\int_{\Gamma_0} \omega = \int_{-1}^{1} \left(4 + \frac{4}{4+4t^2}\right) dt = \int_{-1}^{1} \left(4 + \frac{1}{1+t^2}\right) dt = \big[4t + \arctan t\big]_{-1}^{1} = \left(4 + \frac{\pi}{4}\right) - \left(-4 - \frac{\pi}{4}\right) = 8 + \frac{\pi}{2}$.

Exercise 6

1. Let $D = \{(x,y) \in \mathbb{R}^2 : y \geq 0,\ 1 \leq x^2 + y^2 \leq 3\}$, and let Γ^+ denote its positively oriented boundary. Compute ★★☆☆☆

$$I = \oint_{\Gamma^+} \left(e^{x^2}\sin x - y^3 + 2y\right) dx + \left(x^3 + 4x + \ln(2 + \cos y)\right) dy.$$

Method ***Green–Riemann Formula and Polar Coordinates on an Upper Half-Annulus***

> ***Why we do this (Way of Thinking):***
> Direct parameterization along the four pieces of Γ^+ (two semicircles and two radial segments) is blocked by the terms $e^{x^2}\sin x\,dx$ and $\ln(2+\cos y)\,dy$, which have no elementary primitives along circular arcs. Notice, however, that $e^{x^2}\sin x$ depends solely on x while $\ln(2+\cos y)$ depends solely on y: when we apply the Green–Riemann formula $\oint_{\Gamma^+}(P\,dx+Q\,dy)=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$, both non-elementary terms differentiate to 0 and disappear completely. Even better, the remaining integrand $3(x^2+y^2)+2$ depends only on $r^2=x^2+y^2$, matching the polar rectangle $[1,\sqrt{3}]\times[0,\pi]$ of the upper half-annulus D.

Execution: Step 1 — Green–Riemann reduction on the upper half-annulus D: Define

$$P(x,y)=e^{x^2}\sin x-y^3+2y,\qquad Q(x,y)=x^3+4x+\ln(2+\cos y).$$

Since $\cos y\geq -1$ for all $y\in\mathbb{R}$, we have $2+\cos y\geq 1>0$, so $P,Q\in C^1(\mathbb{R}^2)$. The bounded upper half-annulus D (Figure 2) decomposes into finitely many elementary domains and is bounded by the piecewise C^1 positively oriented curve $\Gamma^+=\partial D^+$ (keeping D on the left).

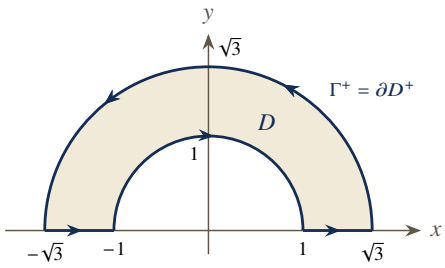


Figure 2: Upper half-annulus $D=\{y\geq 0,\ 1\leq x^2+y^2\leq 3\}$ with positively oriented boundary Γ^+.

Computing the partial derivatives on $\mathbb{R}^2$:

$$\frac{\partial Q}{\partial x}(x,y)=\frac{\partial}{\partial x}\Big(x^3+4x+\ln(2+\cos y)\Big)=3x^2+4+0=3x^2+4,$$
$$\frac{\partial P}{\partial y}(x,y)=\frac{\partial}{\partial y}\Big(e^{x^2}\sin x-y^3+2y\Big)=0-3y^2+2=-3y^2+2.$$

Hence

$$\frac{\partial Q}{\partial x}(x,y)-\frac{\partial P}{\partial y}(x,y)=(3x^2+4)-(-3y^2+2)=3x^2+3y^2+2=3(x^2+y^2)+2.$$

Applying the Green–Riemann formula on D:

$$I=\oint_{\Gamma^+}(P\,dx+Q\,dy)=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy=\iint_D\Big(3(x^2+y^2)+2\Big)\,dx\,dy.$$

Step 2 — Evaluation in polar coordinates:

Sub **Pass to polar coordinates $x=r\cos\theta$, $y=r\sin\theta$ $(r>0)$, with $dx\,dy=r\,dr\,d\theta$. Bounds on D:** $1\leq r^2\leq 3\implies 1\leq r\leq\sqrt{3}$; $y=r\sin\theta\geq 0\implies 0\leq\theta\leq\pi$.

By Fubini's theorem on the polar rectangle $\Delta=[1,\sqrt{3}]\times[0,\pi]$:

$$I=\int_0^{\pi}d\theta\int_1^{\sqrt{3}}(3r^2+2)\,r\,dr=\Big[\theta\Big]_{\theta=0}^{\theta=\pi}\int_1^{\sqrt{3}}(3r^3+2r)\,dr$$
$$=\pi\left[\frac{3}{4}r^4+r^2\right]_{r=1}^{r=\sqrt{3}}=\pi\left[\left(\frac{3}{4}(\sqrt{3})^4+(\sqrt{3})^2\right)-\left(\frac{3}{4}(1)^4+1^2\right)\right]$$
$$=\pi\left[\left(\frac{27}{4}+3\right)-\left(\frac{3}{4}+1\right)\right]=\pi\left(\frac{24}{4}+2\right)=\pi(6+2)=8\pi.$$

$$\boxed{I = 8\pi}$$

> ▸ **Common Pitfall:**
> Two frequent polar bounds errors occur here: (1) writing $1 \le r \le 3$ instead of $1 \le r \le \sqrt{3}$ for $1 \le x^2 + y^2 \le 3$; and (2) integrating θ over $[0, 2\pi]$ instead of $[0, \pi]$, overlooking the condition $y \ge 0$ (which doubles the result to 16π).

> ✓ **Sanity Check:**
> The upper half-annulus has area $\mathcal{A}(D) = \frac{1}{2}\pi\big((\sqrt{3})^2 - 1^2\big) = \pi$. On D, $1 \le x^2 + y^2 \le 3$, so $5 \le 3(x^2 + y^2) + 2 \le 11$. Integrating over $\mathcal{A}(D) = \pi$ requires $5\pi \le I \le 11\pi$, and $I = 8\pi$ lies right at the midpoint $\frac{5\pi+11\pi}{2}$.

2. Fix $a, b > 0$ and let Γ be the upper half-ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $y \ge 0$, oriented from $A(a, 0)$ to $B(-a, 0)$. Compute ★★★☆☆

$$J = \int_{\Gamma} \big(e^x \cos y - 3y\big)\, dx + \big(x - e^x \sin y\big)\, dy.$$

Method *Open-Arc Closure by an Auxiliary Segment and Green–Riemann Formula*

> ***Why we do this (Way of Thinking):***
> The Green–Riemann formula applies strictly to *closed* boundary curves, whereas Γ is an open arc running from $A(a, 0)$ to $B(-a, 0)$. However, $\frac{\partial}{\partial x}(x - e^x \sin y) - \frac{\partial}{\partial y}(e^x \cos y - 3y) = (1 - e^x \sin y) - (-e^x \sin y - 3) = 4$ is constant everywhere on $\mathbb{R}^2$. Closing Γ with the straight segment Σ along the x-axis from $B(-a, 0)$ back to $A(a, 0)$ completes the counterclockwise boundary $\partial E^+ = \Gamma \cup \Sigma$ of the upper half-ellipse E (area $\frac{1}{2}\pi ab$). Subtracting the elementary one-variable integral $\int_{\Sigma} = \int_{-a}^{a} e^x\, dx$ from $\oint_{\partial E^+} = 4\,\mathcal{A}(E)$ isolates J immediately.

Execution: Step 1 — Contour closure $\partial E^+ = \Gamma \cup \Sigma$: Introduce the oriented horizontal line segment Σ from $B(-a, 0)$ to $A(a, 0)$ along the Ox-axis, parameterized by

$$x(t) = t, \qquad y(t) = 0, \qquad t \in [-a, a].$$

As shown in Figure 3, concatenating Γ (from $A(a, 0)$ to $B(-a, 0)$ in $y \ge 0$) with Σ (from $B(-a, 0)$ to $A(a, 0)$ along $y = 0$) produces the positively oriented piecewise C^1 boundary $\partial E^+ = \Gamma \cup \Sigma$ of the upper half-ellipse

$$E = \left\{(x, y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1,\ y \ge 0\right\}.$$

Since the full ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$ has area πab, reflection symmetry across Ox gives $\mathcal{A}(E) = \dfrac{1}{2}\pi ab$.

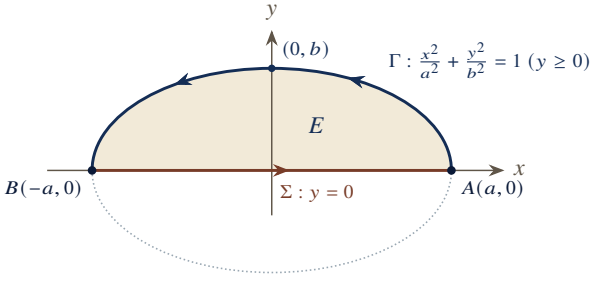


Figure 3: Closing the open upper semi-ellipse Γ with the horizontal segment Σ from $B(-a, 0)$ to $A(a, 0)$ to form $\partial E^+ = \Gamma \cup \Sigma$.

Step 2 — Green–Riemann formula on $\partial E^+ = \Gamma \cup \Sigma$: Set $P(x,y) = e^x \cos y - 3y$ and $Q(x,y) = x - e^x \sin y$. Both P and Q belong to $C^1(\mathbb{R}^2)$, and on $\mathbb{R}^2$:

$$\frac{\partial Q}{\partial x}(x,y) = \frac{\partial}{\partial x}\big(x - e^x \sin y\big) = 1 - e^x \sin y,$$
$$\frac{\partial P}{\partial y}(x,y) = \frac{\partial}{\partial y}\big(e^x \cos y - 3y\big) = -e^x \sin y - 3,$$

whence

$$\frac{\partial Q}{\partial x}(x,y) - \frac{\partial P}{\partial y}(x,y) = \big(1 - e^x \sin y\big) - \big(-e^x \sin y - 3\big) = 4.$$

Applying the Green–Riemann formula on the elementary domain E and using additivity along $\partial E^+ = \Gamma \cup \Sigma$:

$$J + \int_\Sigma \big(P\,dx + Q\,dy\big) = \oint_{\partial E^+} \big(P\,dx + Q\,dy\big) = \iint_E 4\,dx\,dy = 4\,\mathcal{A}(E) = 4 \cdot \frac{1}{2}\pi ab = 2\pi ab.$$

Step 3 — Line integral along Σ and deduction of J: Along Σ, $x(t) = t \in [-a, a]$ and $y(t) = 0$, so $\boldsymbol{dx = dt}$ and $\boldsymbol{dy = 0}$. Substituting into $\int_\Sigma (P\,dx + Q\,dy)$:

$$\begin{aligned}\int_\Sigma \big(e^x \cos y - 3y\big)\,dx + \big(x - e^x \sin y\big)\,dy &= \int_{-a}^{a} \Big(\big(e^t \cos 0 - 3 \cdot 0\big) \cdot 1 + \big(t - e^t \sin 0\big) \cdot 0\Big)\,dt \\ &= \int_{-a}^{a} e^t\,dt = \Big[e^t\Big]_{t=-a}^{t=a} = e^a - e^{-a} = 2\sinh a.\end{aligned}$$

Subtracting $\int_\Sigma (P\,dx + Q\,dy)$ from $2\pi ab$ gives

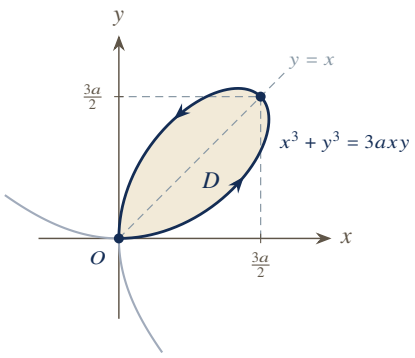
$$J = 2\pi ab - \int_\Sigma \big(P\,dx + Q\,dy\big) = 2\pi ab - \big(e^a - e^{-a}\big) = 2\pi ab - 2\sinh a.$$

$$\boxed{J = 2\pi ab - \big(e^a - e^{-a}\big) = 2\pi ab - 2\sinh a}$$

> ► **Common Pitfall:**
> Writing $J = \iint_E 4\,dx\,dy = 2\pi ab$ directly forgets that Γ is an *open* arc: even though $y = 0$ and $dy = 0$ on the closing segment Σ, the term $P(x,0)\,dx = e^x\,dx$ does not vanish along Σ. Also keep the orientation straight: Σ runs from $B(-a,0)$ to $A(a,0)$, so $J + \int_\Sigma = 2\pi ab \implies J = 2\pi ab - \int_\Sigma$.

> ✓ **Sanity Check:**
> Cross-check without Green–Riemann by grouping $e^x \cos y\,dx - e^x \sin y\,dy = d(e^x \cos y)$, which is exact on $\mathbb{R}^2$: its integral from $A(a,0)$ to $B(-a,0)$ is $e^{-a}\cos 0 - e^a \cos 0 = -2\sinh a$. Parameterizing Γ by $x = a\cos t$, $y = b \sin t$ $(t \in [0,\pi])$ gives $\int_\Gamma (-3y\,dx + x\,dy) = \int_0^\pi \big(3ab\sin^2 t + ab\cos^2 t\big)\,dt = ab\left(\frac{3\pi}{2} + \frac{\pi}{2}\right) = 2\pi ab$, whose sum with $-2\sinh a$ is $2\pi ab - 2\sinh a$.

3. Fix $a > 0$ and let $D \subset \mathbb{R}^2$ be the domain bounded by the loop of the folium of Descartes $x^3 + y^3 = 3axy$ $(x \geq 0,\ y \geq 0)$ represented below.



(a) Setting $y = tx$ ($t \geq 0$), parameterize the loop for $(x, y) \neq (0, 0)$ and specify the orientation induced as t increases. ★★★☆☆

Method ***Rational Slope Parameterization ($y = tx$) and Polar-Angle Monotonicity***

> ***Why we do this (Way of Thinking):***
> The cubic equation $x^3 + y^3 = 3axy$ cannot be solved simply as $y = f(x)$, yet its left-hand side is homogeneous of degree 3 while its right-hand side is homogeneous of degree 2. Intersecting the loop with the rotating ray $y = tx$ ($t = \frac{y}{x}$ being the slope from the origin) factors out x^2 and expresses both $x(t)$ and $y(t)$ rationally in t. Because $t = \frac{y}{x} = \tan\theta$ is the tangent of the polar angle $\theta \in \left]0, \frac{\pi}{2}\right[$, increasing t from 0 to $+\infty$ increases θ strictly from 0 to $\frac{\pi}{2}$, sweeping around D counterclockwise.

Execution: Step 1 — Rational parameterization of the first-quadrant loop: On $x^3 + y^3 = 3axy$, setting $x = 0$ yields $y^3 = 0 \implies y = 0$, and setting $y = 0$ yields $x^3 = 0 \implies x = 0$. Therefore, every point $(x, y) \neq (0, 0)$ on the loop in the first quadrant ($x \geq 0, y \geq 0$) satisfies $x > 0$ and $y > 0$ strictly. Substituting $y = tx$ with $t = \dfrac{y}{x} > 0$ into $x^3 + y^3 = 3axy$ gives

$$x^3 + (tx)^3 = 3ax(tx) \iff x^3(1 + t^3) = 3atx^2.$$

Since $x > 0$, we have $x^2 > 0$ and $1 + t^3 > 1 > 0$. Dividing both sides by $x^2(1 + t^3)$ isolates $x(t)$, and $y(t) = t\,x(t)$ follows:

$$x(t) = \frac{3at}{1 + t^3}, \qquad y(t) = \frac{3at^2}{1 + t^3}, \qquad t > 0.$$

Including $t = 0$ recovers the departure from the origin $\big(x(0), y(0)\big) = (0, 0)$, while at $t = 1$ the curve passes through the vertex $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ on $y = x$, and as $t \to +\infty$,

$$\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} \frac{3a}{t^{-1} + t^2} = 0, \qquad \lim_{t\to+\infty} y(t) = \lim_{t\to+\infty} \frac{3a}{t^{-2} + t} = 0,$$

returning asymptotically in parameter t to $(0, 0)$.

Step 2 — Induced orientation on ∂D: For every $t > 0$, $(x(t), y(t))$ lies in the open first quadrant ($x(t) > 0, y(t) > 0$), so its polar angle $\theta(t) \in \left]0, \frac{\pi}{2}\right[$ satisfies

$$\tan\theta(t) = \frac{y(t)}{x(t)} = t \quad \implies \quad \theta(t) = \arctan t, \qquad \theta'(t) = \frac{1}{1 + t^2} > 0.$$

As t increases from 0 to $+\infty$, $\theta(t)$ increases strictly from 0 (tangent to $+Ox$ along the bottom of D) through $\frac{\pi}{4}$ up to $\frac{\pi}{2}$ (tangent to $+Oy$ along the left side of D), keeping the domain D on the left throughout. Thus increasing t induces the **positive (counterclockwise) orientation** on ∂D^+.

$$\gamma(t) = \left(\frac{3at}{1 + t^3}, \frac{3at^2}{1 + t^3}\right), \quad t \in]0, +\infty[\quad \big(\text{with } \gamma(0) = (0, 0) \text{ and } \lim_{t\to+\infty} \gamma(t) = (0, 0), \text{ CCW}\big)$$

> ▸ **Common Pitfall:**
> Stopping the parameter range at $t = 1$ covers only the *lower* half of the loop from $(0, 0)$ to $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ ($0 \leq y \leq x$). The upper half ($0 \leq x \leq y$) from $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ back to $(0, 0)$ requires $t \in [1, +\infty[$.

> ✓ **Sanity Check:**
> Under the inversion $t \mapsto \frac{1}{t}$ ($t > 0$), $x\left(\frac{1}{t}\right) = \frac{3a/t}{1+1/t^3} = \frac{3at^2}{t^3+1} = y(t)$ and $y\left(\frac{1}{t}\right) = x(t)$, reflecting the symmetry of $x^3 + y^3 = 3axy$ across the bisector $y = x$.

(b) Compute the area $\mathcal{A}(D)$. ★★★★☆

Method ***Green–Riemann Planar Area Formula*** ($\frac{1}{2}\oint_{\partial D^+}(x\,dy - y\,dx)$)***, Product-Rule Cancellation, and Substitution*** $u = 1 + t^3$

> ***Why we do this (Way of Thinking):***
> By the Green–Riemann area corollary, $\mathcal{A}(D) = \frac{1}{2}\oint_{\partial D^+}(x\,dy - y\,dx)$. Rather than expanding $x'(t)$ and $y'(t)$ separately by the quotient rule, exploit the relation $y(t) = t\,x(t)$: the product rule gives $y'(t) = x(t) + t\,x'(t)$, so in $x(t)y'(t) - y(t)x'(t)$ the derivative $x'(t)$ cancels out identically, leaving $\big(x(t)\big)^2 = \frac{9a^2t^2}{(1+t^3)^2}$. To justify the unbounded interval $t \in [0, +\infty[$ rigorously, truncate at $t = M > 0$ and close the sector D_M back to $(0,0)$ along the ray $y = Mx$ (where $x\,dy - y\,dx = 0$), then let $M \to +\infty$ and substitute $u = 1 + t^3$.

Execution: Step 1 — Simplification of $x(t)y'(t) - y(t)x'(t)$: For all $t \geq 0$, differentiating $y(t) = t\,x(t)$ by the product rule gives $y'(t) = x(t) + t\,x'(t)$. Therefore:

$$\begin{aligned} x(t)y'(t) - y(t)x'(t) &= x(t)\big(x(t) + t\,x'(t)\big) - \big(t\,x(t)\big)x'(t) \\ &= \big(x(t)\big)^2 + t\,x(t)x'(t) - t\,x(t)x'(t) = \big(x(t)\big)^2 = \left(\frac{3at}{1+t^3}\right)^2 = \frac{9a^2t^2}{(1+t^3)^2}. \end{aligned}$$

Step 2 — Green–Riemann area integral via sector exhaustion: For each $M > 0$, let $D_M = \big\{(x,y) \in D : y \leq Mx\big\}$ be the compact subdomain bounded counterclockwise by the folium arc $\Gamma_M : \gamma(t)$ $(t \in [0, M])$ from $(0,0)$ to $\gamma(M)$ and the straight radial segment L_M from $\gamma(M)$ back to $(0,0)$ along the line $y = Mx$. Along L_M, $y = Mx \implies dy = M\,dx$, hence

$$x\,dy - y\,dx = x(M\,dx) - (Mx)\,dx = 0 \quad\Longrightarrow\quad \int_{L_M}(x\,dy - y\,dx) = 0.$$

Applying $\mathcal{A}(D_M) = \dfrac{1}{2}\oint_{\partial D_M^+}(x\,dy - y\,dx)$ on the compact piecewise C^1 boundary $\partial D_M^+ = \Gamma_M \cup L_M$:

$$\begin{aligned} \mathcal{A}(D_M) &= \frac{1}{2}\int_{\Gamma_M}(x\,dy - y\,dx) + \frac{1}{2}\int_{L_M}(x\,dy - y\,dx) \\ &= \frac{1}{2}\int_0^M \big(x(t)y'(t) - y(t)x'(t)\big)\,dt = \frac{9a^2}{2}\int_0^M \frac{t^2}{(1+t^3)^2}\,dt. \end{aligned}$$

Letting $M \to +\infty$ exhausts $D \setminus \{(0,0)\}$ (and $\{(0,0)\}$ has zero area), yielding

$$\mathcal{A}(D) = \lim_{M\to+\infty}\mathcal{A}(D_M) = \frac{9a^2}{2}\int_0^{+\infty}\frac{t^2}{(1+t^3)^2}\,dt.$$

Step 3 — Explicit evaluation via $u = 1 + t^3$:

Sub **Let $u = 1 + t^3 \implies du = 3t^2\,dt \implies t^2\,dt = \frac{1}{3}\,du$. Bounds: $t = 0 \implies u = 1$; $t \to +\infty \implies u \to +\infty$.**

Evaluating the improper integral step by step:

$$\begin{aligned} \mathcal{A}(D) &= \frac{9a^2}{2}\int_1^{+\infty}\frac{1}{u^2}\cdot\frac{1}{3}\,du = \frac{3a^2}{2}\int_1^{+\infty}u^{-2}\,du \\ &= \frac{3a^2}{2}\lim_{U\to+\infty}\left[-\frac{1}{u}\right]_{u=1}^{u=U} = \frac{3a^2}{2}\lim_{U\to+\infty}\left(-\frac{1}{U} + 1\right) = \frac{3a^2}{2}(0+1) = \frac{3a^2}{2}. \end{aligned}$$

$$\boxed{\mathcal{A}(D) = \frac{3a^2}{2}}$$

> **► Common Pitfall:**
> Omitting the factor $\frac{1}{2}$ in $\mathcal{A}(D) = \frac{1}{2}\oint_{\partial D^+}(x\,dy - y\,dx)$ (which compensates for $\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y} = 1-(-1) = 2$) doubles the area to $3a^2$, while writing $y\,dx - x\,dy$ instead of $x\,dy - y\,dx$ reverses the sign to $-\frac{3a^2}{2}$.

> **✓ Sanity Check:**
> By symmetry across the bisector $y = x$, $\mathcal{A}(D) = 2\,\mathcal{A}(D_1)$ where $D_1 = D \cap \{y \le x\}$ corresponds to the compact interval $t \in [0,1]$ closed along $y = x$ ($x\,dy - y\,dx = 0$): $\mathcal{A}(D) = 2\cdot\frac{9a^2}{2}\int_0^1 \frac{t^2}{(1+t^3)^2}\,dt = 3a^2\left[-\frac{1}{1+t^3}\right]_0^1 = 3a^2\left(1-\frac{1}{2}\right) = \frac{3a^2}{2}$, avoiding improper limits altogether and confirming the exact constant $\frac{3}{2}$.

Exercise 7

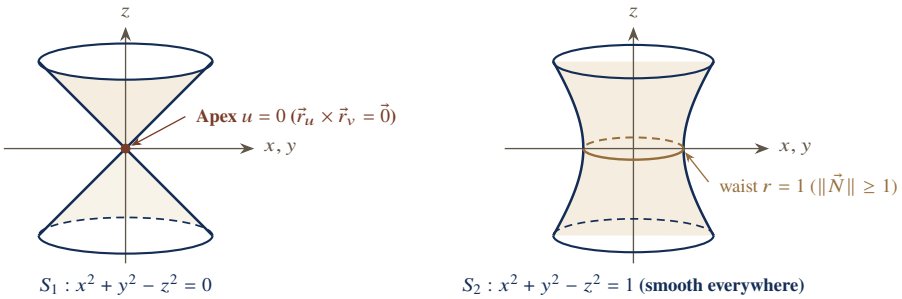


Figure 4: Comparison between the double circular cone S_1 (non-smooth at its apex $u = 0$) and the ruled one-sheeted hyperboloid S_2 (smooth everywhere since $\|\vec{N}(u,v)\|^2 = 2u^2 + 1 \ge 1$).

1. Consider the parameterized surface S_1 given by

$$\vec{r}(u,v) = u\cos v\,\vec{i} + u\sin v\,\vec{j} + u\,\vec{k}, \qquad (u,v) \in \mathbb{R}\times[0,2\pi].$$

Find a Cartesian equation satisfied by the points of S_1, compute the fundamental cross product $\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v}$, and determine all pairs $(u,v) \in \mathbb{R}\times[0,2\pi]$ at which S_1 fails to be smooth. ★★☆☆☆

Method ***Trigonometric Parameter Elimination, Partial Velocity Cross Product, and Smoothness Criterion***

> ***Why we do this (Way of Thinking):***
> To eliminate (u,v) from $x = u\cos v$, $y = u\sin v$, $z = u$, square and add x and y so that $\cos^2 v + \sin^2 v = 1$ eliminates v and leaves $x^2 + y^2 = u^2 = z^2$. Next, recall from the course that a C^1 parameterized surface is smooth where the fundamental normal cross product $\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v}$ is nonzero, and fails to be smooth where $\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v} = \vec{0}$ (meaning the two coordinate tangent vectors cease to span a 2-dimensional tangent plane).

Execution: Step 1 — Cartesian equation of S_1: Writing $x = u\cos v$, $y = u\sin v$, and $z = u$ for $(u,v) \in \mathbb{R}\times[0,2\pi]$, we have

$$x^2 + y^2 = (u\cos v)^2 + (u\sin v)^2 = u^2(\cos^2 v + \sin^2 v) = u^2 = z^2.$$

Hence every point (x,y,z) of S_1 satisfies the Cartesian equation

$$x^2 + y^2 - z^2 = 0,$$

which describes the double circular cone of revolution around Oz with apex at $(0,0,0)$ (Figure 4, left).

Step 2 — Fundamental cross product $\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$: Differentiating $\vec{r} \in C^\infty(\mathbb{R}^2, \mathbb{R}^3)$ with respect to u and v:

$$\frac{\partial \vec{r}}{\partial u}(u,v) = \cos v\,\vec{i} + \sin v\,\vec{j} + 1\,\vec{k}, \qquad \frac{\partial \vec{r}}{\partial v}(u,v) = -u\sin v\,\vec{i} + u\cos v\,\vec{j} + 0\,\vec{k}.$$

Expanding the 3×3 determinant along the top row:

$$\begin{aligned}\frac{\partial \vec{r}}{\partial u}(u,v) \times \frac{\partial \vec{r}}{\partial v}(u,v) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{vmatrix} \\ &= \big(\sin v \cdot 0 - 1 \cdot u\cos v\big)\vec{i} - \big(\cos v \cdot 0 - 1 \cdot (-u\sin v)\big)\vec{j} + \big(u\cos^2 v - (-u\sin^2 v)\big)\vec{k} \\ &= -u\cos v\,\vec{i} - u\sin v\,\vec{j} + u(\cos^2 v + \sin^2 v)\,\vec{k} \\ &= -u\cos v\,\vec{i} - u\sin v\,\vec{j} + u\,\vec{k}.\end{aligned}$$

Step 3 — Parameter pairs where smoothness fails: Computing the squared Euclidean norm of the cross product:

$$\left\| \frac{\partial \vec{r}}{\partial u}(u,v) \times \frac{\partial \vec{r}}{\partial v}(u,v) \right\|^2 = (-u\cos v)^2 + (-u\sin v)^2 + u^2 = u^2(\cos^2 v + \sin^2 v) + u^2 = 2u^2.$$

Therefore, for $(u,v) \in \mathbb{R} \times [0, 2\pi]$,

$$\frac{\partial \vec{r}}{\partial u}(u,v) \times \frac{\partial \vec{r}}{\partial v}(u,v) = \vec{0} \iff 2u^2 = 0 \iff u = 0 \quad (\text{with } v \in [0,2\pi] \text{ arbitrary}).$$

Thus S_1 fails to be smooth at all pairs in $\{(0,v) : v \in [0,2\pi]\}$, all of which map to the conical apex $\vec{r}(0,v) = (0,0,0)$.

> Cartesian equation: $x^2 + y^2 - z^2 = 0$; $\quad \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = -u\cos v\,\vec{i} - u\sin v\,\vec{j} + u\,\vec{k}$;
> Non-smooth parameter pairs: $\{(0,v) : v \in [0,2\pi]\}$

► Common Pitfall:
Two frequent errors occur here: (1) forgetting the cofactor minus sign on the $\vec{j}$-component of the 3×3 determinant (writing $+u\sin v\,\vec{j}$ instead of $-u\sin v\,\vec{j}$); and (2) reporting only $(0,0)$ instead of the entire segment $\{(0,v) : v \in [0,2\pi]\}$ in the (u,v)-domain.

✓ Sanity Check:
Verify orthogonality to both coordinate tangent vectors: $(-u\cos v, -u\sin v, u) \cdot (\cos v, \sin v, 1) = -u(\cos^2 v + \sin^2 v) + u = 0$, and $(-u\cos v, -u\sin v, u) \cdot (-u\sin v, u\cos v, 0) = u^2 \sin v\cos v - u^2 \sin v\cos v = 0$.

2. Consider the parameterized surface S_2 defined by

$$\vec{\Phi}(u,v) = (u\cos v - \sin v)\,\vec{i} + (u\sin v + \cos v)\,\vec{j} + u\,\vec{k}, \qquad (u,v) \in \mathbb{R} \times [0, 2\pi[.$$

(a) Verify that every point (x,y,z) of S_2 lies on the quadric $x^2+y^2-z^2=1$, compute $\vec{N}(u,v) = \frac{\partial \vec{\Phi}}{\partial u} \times \frac{\partial \vec{\Phi}}{\partial v}$, and prove that S_2 is smooth everywhere on $\mathbb{R} \times [0, 2\pi[$. ★★☆☆☆

Method *Algebraic Quadric Verification, Fundamental Normal Cross Product, and Strict Positivity*

of $\|\vec{N}\|^2$

> ***Why we do this (Way of Thinking):***
> Writing $\vec{\Phi}(u,v) = (-\sin v, \cos v, 0) + u(\cos v, \sin v, 1)$ shifts the rulings of the cone S_1 outward so that at $u = 0$ they pass through the unit waist circle $(-\sin v, \cos v, 0)$ instead of pinching together at the origin (Figure 4, right). Expanding $x^2 + y^2 - z^2$ cancels the cross terms $\pm 2u \sin v \cos v$. To prove global smoothness cleanly without solving coupled trigonometric equations, compute $\|\vec{N}(u,v)\|^2 = 2u^2 + 1 \geq 1 > 0$.

Execution: Step 1 — Quadric identity $x^2 + y^2 - z^2 = 1$**:** Setting $x = u\cos v - \sin v$, $y = u \sin v + \cos v$, and $z = u$, expand $x^2 + y^2 - z^2$ step by step:

$$\begin{aligned} x^2 + y^2 - z^2 &= (u\cos v - \sin v)^2 + (u \sin v + \cos v)^2 - u^2 \\ &= \big(u^2\cos^2 v - 2u\cos v \sin v + \sin^2 v\big) + \big(u^2 \sin^2 v + 2u \sin v \cos v + \cos^2 v\big) - u^2 \\ &= u^2(\cos^2 v + \sin^2 v) + (\sin^2 v + \cos^2 v) - u^2 \\ &= u^2(1) + 1 - u^2 = 1. \end{aligned}$$

Thus every point of S_2 lies on the quadric $x^2 + y^2 - z^2 = 1$.

Step 2 — Computation of $\vec{N}(u,v) = \dfrac{\partial\vec{\Phi}}{\partial u} \times \dfrac{\partial\vec{\Phi}}{\partial v}$**:** Differentiating $\vec{\Phi} \in C^\infty(\mathbb{R}^2, \mathbb{R}^3)$ with respect to u and v:

$$\frac{\partial\vec{\Phi}}{\partial u}(u,v) = \cos v\,\vec{i} + \sin v\,\vec{j} + 1\,\vec{k}, \qquad \frac{\partial\vec{\Phi}}{\partial v}(u,v) = (-u\sin v - \cos v)\,\vec{i} + (u\cos v - \sin v)\,\vec{j} + 0\,\vec{k}.$$

Expanding the cross-product determinant:

$$\begin{aligned} \vec{N}(u,v) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 1 \\ -u\sin v - \cos v & u\cos v - \sin v & 0 \end{vmatrix} \\ &= \big(0 - (u\cos v - \sin v)\big)\vec{i} - \big(0 - (-u\sin v - \cos v)\big)\vec{j} \\ &\quad + \Big(\cos v(u\cos v - \sin v) - \sin v(-u\sin v - \cos v)\Big)\vec{k} \\ &= (\sin v - u\cos v)\,\vec{i} - (u\sin v + \cos v)\,\vec{j} + \big(u\cos^2 v - \sin v\cos v + u\sin^2 v + \sin v\cos v\big)\vec{k} \\ &= (\sin v - u\cos v)\,\vec{i} - (u\sin v + \cos v)\,\vec{j} + u\,\vec{k}. \end{aligned}$$

Step 3 — Smoothness everywhere on $\mathbb{R}\times[0, 2\pi[$**:** Observe that $\vec{N}(u,v) = -x(u,v)\,\vec{i} - y(u,v)\,\vec{j} + z(u,v)\,\vec{k}$. Using $x^2 + y^2 = 1 + z^2 = 1 + u^2$ from Step 1:

$$\begin{aligned} \|\vec{N}(u,v)\|^2 &= (\sin v - u\cos v)^2 + \big(-(u\sin v + \cos v)\big)^2 + u^2 \\ &= (u^2 + 1) + u^2 = 2u^2 + 1 \geq 1 > 0 \qquad \forall (u,v) \in \mathbb{R}\times[0, 2\pi[. \end{aligned}$$

Hence $\vec{N}(u,v) \neq \vec{0}$ for every $(u,v) \in \mathbb{R}\times[0, 2\pi[$, proving that S_2 is smooth everywhere on $\mathbb{R}\times[0, 2\pi[$.

$$\boxed{\vec{N}(u,v) = (\sin v - u\cos v)\,\vec{i} - (u\sin v + \cos v)\,\vec{j} + u\,\vec{k}, \qquad \|\vec{N}(u,v)\|^2 = 2u^2 + 1 \geq 1 > 0}$$

> ► **Common Pitfall:**
> In the $\vec{j}$-cofactor of $\vec{N}(u,v)$, the $(1,2)$ minor equals $\cos v \cdot 0 - 1\cdot(-u\sin v - \cos v) = +(u\sin v + \cos v)$, which is then negated by the $(-1)^{1+2} = -1$ cofactor sign to give $-(u\sin v + \cos v)\,\vec{j}$.

> ✓ **Sanity Check:**
> Since S_2 lies on the level surface $G(x,y,z) = x^2 + y^2 - z^2 - 1 = 0$, the parametric normal $\vec{N}(u,v)$ must be collinear with $\nabla G(x,y,z) = (2x, 2y, -2z)$. Indeed, $\vec{N}(u,v) = (-x, -y, z) = -\frac{1}{2}\nabla G\big(\vec{\Phi}(u,v)\big)$, verifying all three components and signs simultaneously.

(b) Determine the Cartesian equation of the tangent plane to S_2 at the point corresponding to $(u_0, v_0) = (2, 0)$. ★★☆☆☆

Method ***Cartesian Tangent Plane Formula*** $(M - M_0) \cdot \vec{N}(u_0, v_0) = 0$

> ***Why we do this (Way of Thinking):***
> At any smooth parameter pair (u_0, v_0), the tangent plane passes through the surface point $M_0 = \vec{\Phi}(u_0, v_0)$ and is orthogonal to the normal vector $\vec{N}(u_0, v_0)$ already derived in Question 2(a). Evaluating both vectors at $(2, 0)$ and expanding $(x - x_0, y - y_0, z - z_0) \cdot \vec{N}(2, 0) = 0$ gives the Cartesian equation directly.

Execution: Step 1 — Base point M_0 and normal vector $\vec{N}(2, 0)$: Substituting $u_0 = 2$ and $v_0 = 0$ ($\cos 0 = 1$, $\sin 0 = 0$) into $\vec{\Phi}(u, v)$ and $\vec{N}(u, v)$:

$$M_0 = \vec{\Phi}(2, 0) = (2 \cdot 1 - 0,\ 2 \cdot 0 + 1,\ 2) = (2, 1, 2),$$

$$\vec{N}(2, 0) = (0 - 2 \cdot 1)\,\vec{i} - (2 \cdot 0 + 1)\,\vec{j} + 2\,\vec{k} = -2\,\vec{i} - \vec{j} + 2\,\vec{k}.$$

Step 2 — Cartesian equation of the tangent plane: Applying $(x - x_0, y - y_0, z - z_0) \cdot \vec{N}(2, 0) = 0$:

$$\begin{aligned} -2(x - 2) - 1(y - 1) + 2(z - 2) = 0 &\iff -2x + 4 - y + 1 + 2z - 4 = 0 \\ &\iff -2x - y + 2z + 1 = 0 \\ &\iff 2x + y - 2z = 1. \end{aligned}$$

$$\boxed{2x + y - 2z = 1}$$

> ▸ **Common Pitfall:**
> Do not confuse the parameter pair $(u_0, v_0) = (2, 0) \in \mathbb{R}^2$ with the spatial coordinates of M_0: at $v_0 = 0$, $y_0 = u_0 \sin 0 + \cos 0 = 1 \neq 0$, so $M_0 = (2, 1, 2)$.

> ✓ **Sanity Check:**
> For fixed $v = 0$, the straight ruling line $u \mapsto \vec{\Phi}(u, 0) = (u, 1, u)$ lies entirely on S_2 and passes through $M_0(2, 1, 2)$, so it must lie completely inside the tangent plane at M_0. Indeed, $2(u) + 1 - 2(u) = 1$ holds identically for all $u \in \mathbb{R}$.

(c) Determine all pairs $(u, v) \in \mathbb{R} \times [0, 2\pi[$ at which the tangent plane to S_2 is parallel to the plane $x + y - z = 0$, and write the Cartesian equations of the corresponding tangent planes. ★★★☆☆

Method ***Normal-Vector Collinearity ($\vec{N}(u, v) = \lambda\,\vec{n}_0$) and Trigonometric System Resolution***

> ***Why we do this (Way of Thinking):***
> Two planes in $\mathbb{R}^3$ are parallel if and only if their normal vectors are nonzero scalar multiples of each other: $\vec{N}(u, v) = \lambda\,\vec{n}_0$ for some $\lambda \in \mathbb{R}^*$, where $\vec{n}_0 = (1, 1, -1)$ is normal to $x + y - z = 0$. Comparing the third components gives $u = -\lambda$ immediately, and using $\|\vec{N}\|^2$ (or the quadric relation $x^2 + y^2 - z^2 = 1$ with $\vec{N} = (-x, -y, z)$) restricts λ to ± 1, reducing (u, v) to two linear systems in $(\cos v, \sin v)$ on $[0, 2\pi[$.

Execution: Step 1 — Collinearity system: The reference plane $x + y - z = 0$ has normal vector $\vec{n}_0 = (1, 1, -1)$. Since $\vec{N}(u, v) \neq \vec{0}$ everywhere on $\mathbb{R} \times [0, 2\pi[$, the tangent plane to S_2 at (u, v) is parallel to $x + y - z = 0$ if and only if there exists $\lambda \in \mathbb{R}^*$ such that $\vec{N}(u, v) = \lambda\,\vec{n}_0$:

$$\begin{cases} \sin v - u \cos v = \lambda, & (1) \\ -u \sin v - \cos v = \lambda, & (2) \\ u = -\lambda. & (3) \end{cases}$$

Step 2 — Solving for λ and u: Squaring and adding (1) and (2) (or equating $\|\vec{N}(u,v)\|^2 = 2u^2+1$ to $\|\lambda\,\vec{n}_0\|^2 = 3\lambda^2$ with $u=-\lambda$):

$$2u^2+1=3\lambda^2 \quad \text{and} \quad u=-\lambda \quad \Longrightarrow \quad 2\lambda^2+1=3\lambda^2 \iff \lambda^2=1 \iff \lambda\in\{-1,1\}.$$

Step 3 — Solving for $v\in[0,2\pi[$ and writing the tangent planes:

- **Case 1 ($\lambda=-1 \implies u=1$):** Equations (1) and (2) become

$$\begin{cases}\sin v-\cos v=-1,\\ -\sin v-\cos v=-1.\end{cases}$$

Adding the two equations gives $-2\cos v=-2 \implies \cos v=1$, and subtracting the second from the first gives $2\sin v=0 \implies \sin v=0$. On $[0,2\pi[$, the unique solution is $v=0$, giving $(u_1,v_1)=(1,0)$ and surface point

$$M_1=\vec{\Phi}(1,0)=(1\cdot 1-0,\ 1\cdot 0+1,\ 1)=(1,1,1).$$

With normal $\vec{n}_0=(1,1,-1)$, the tangent plane at $M_1(1,1,1)$ is

$$1(x-1)+1(y-1)-1(z-1)=0 \iff x+y-z=1.$$

- **Case 2 ($\lambda=1 \implies u=-1$):** Equations (1) and (2) become

$$\begin{cases}\sin v+\cos v=1,\\ \sin v-\cos v=1.\end{cases}$$

Adding the two equations gives $2\sin v=2 \implies \sin v=1$, and subtracting gives $2\cos v=0 \implies \cos v=0$. On $[0,2\pi[$, the unique solution is $v=\dfrac{\pi}{2}$, giving $(u_2,v_2)=\left(-1,\dfrac{\pi}{2}\right)$ and surface point

$$M_2=\vec{\Phi}\left(-1,\frac{\pi}{2}\right)=(-1\cdot 0-1,\ -1\cdot 1+0,\ -1)=(-1,-1,-1).$$

With normal $\vec{n}_0=(1,1,-1)$, the tangent plane at $M_2(-1,-1,-1)$ is

$$1(x+1)+1(y+1)-1(z+1)=0 \iff x+y-z=-1.$$

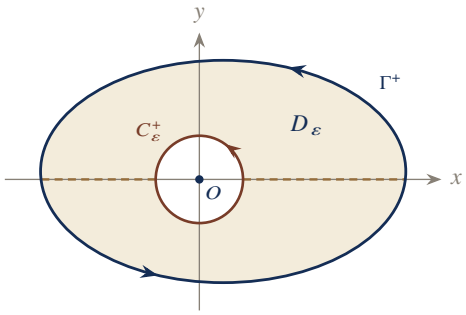
$$\begin{aligned}(u_1,v_1)&=(1,0) &&\longrightarrow \quad \Pi_1: x+y-z=1,\\ (u_2,v_2)&=\left(-1,\frac{\pi}{2}\right) &&\longrightarrow \quad \Pi_2: x+y-z=-1\end{aligned}$$

► Common Pitfall:
Equating $\vec{N}(u,v)=\vec{n}_0$ (forcing $\lambda=1$ instead of $\lambda\in\mathbb{R}^*$) misses the antipodal solution $(u_1,v_1)=(1,0)$, where $\vec{N}(1,0)=(-1,-1,1)=-\vec{n}_0$ still defines the exact same parallel plane direction. Also note that $v=2\pi$ is excluded because $v\in[0,2\pi[$.

✓ Sanity Check:
Since the hyperboloid $x^2+y^2-z^2=1$ is centrally symmetric under $(x,y,z)\mapsto(-x,-y,-z)$, tangent planes parallel to $x+y-z=0$ must occur in centrally symmetric pairs at antipodal points $M_2=-M_1$; indeed $M_1=(1,1,1)$, $M_2=(-1,-1,-1)$, and the planes $x+y-z=\pm 1$ are reflections of each other through the origin.

Exercise 8

Let $D \subset \mathbb{R}^2$ be a bounded domain containing $(0,0)$ in its interior, bounded by a positively oriented piecewise C^1 simple closed curve Γ^+. Fix $\varepsilon > 0$ small enough so that the closed disk $x^2+y^2 \le \varepsilon^2$ lies inside D, set $D_\varepsilon = \{(x,y) \in D : x^2+y^2 \ge \varepsilon^2\}$, and let C_ε^+ denote the circle $x^2+y^2=\varepsilon^2$ oriented counterclockwise (see figure below).



1. Let $P, Q \in C^1(\mathbb{R}^2 \setminus \{(0,0)\})$. Prove that

★★★☆☆

$$\oint_{\Gamma^+} (P\,dx + Q\,dy) - \oint_{C_\varepsilon^+} (P\,dx + Q\,dy) = \iint_{D_\varepsilon} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

Method *Crosscut Domain Decomposition and Opposite-Orientation Cancellation in Green–Riemann*

> ***Why we do this (Way of Thinking):***
> The perforated domain D_ε has an inner circular hole, so its boundary consists of two loops (Γ outside and C_ε inside). Why does $-\oint_{C_\varepsilon^+}$ appear with a minus sign? Walking along ∂D_ε while keeping the shaded region D_ε on the left traverses the outer loop Γ counterclockwise (Γ^+), but traverses the inner hole boundary C_ε *clockwise* ($C_\varepsilon^- = -C_\varepsilon^+$). To deduce this rigorously from the standard Green–Riemann formula on hole-free domains, slice D_ε along the horizontal x-axis (the dashed tan crosscuts in the figure) into upper and lower subdomains D_ε^+ and D_ε^-: each crosscut is traversed from left to right along $\partial(D_\varepsilon^+)^+$ and from right to left along $\partial(D_\varepsilon^-)^+$, so the crosscut integrals cancel identically upon addition.

Execution: Step 1 — C^1 regularity on D_ε and horizontal crosscut decomposition: Because every $(x,y) \in D_\varepsilon$ satisfies $x^2+y^2 \ge \varepsilon^2 > 0$, the compact domain D_ε is contained inside the open set $U_{\varepsilon/2} = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 > \frac{\varepsilon^2}{4}\} \subset \mathbb{R}^2 \setminus \{(0,0)\}$, on which P and Q are of class C^1.

Slice D_ε along the line $y=0$ (the dashed crosscuts Σ_L on $x<0$ and Σ_R on $x>0$, both oriented left-to-right) into the upper half-domain $D_\varepsilon^+ = D_\varepsilon \cap \{y \ge 0\}$ and the lower half-domain $D_\varepsilon^- = D_\varepsilon \cap \{y \le 0\}$ (subdividing further into finitely many elementary domains if Γ^+ crosses $y=0$ more than twice; every interior cut segment cancels in opposite pairs). Split the two closed curves at $y=0$:

- $\Gamma^+ = \Gamma_{\text{top}} \cup \Gamma_{\text{bot}}$, where $\Gamma_{\text{top}} = \Gamma^+ \cap \{y \ge 0\}$ and $\Gamma_{\text{bot}} = \Gamma^+ \cap \{y \le 0\}$ inherit the counterclockwise orientation of Γ^+;
- $C_\varepsilon^+ = C_{\varepsilon,\text{top}}^+ \cup C_{\varepsilon,\text{bot}}^+$, where $C_{\varepsilon,\text{top}}^+$ runs from $(\varepsilon,0)$ to $(-\varepsilon,0)$ in $y \ge 0$ and $C_{\varepsilon,\text{bot}}^+$ runs from $(-\varepsilon,0)$ to $(\varepsilon,0)$ in $y \le 0$, while $C_{\varepsilon,\text{top}}^-$ and $C_{\varepsilon,\text{bot}}^-$ denote the same semicircles with opposite (clockwise) orientation.

Step 2 — Green–Riemann on D_ε^+ and D_ε^-: Set $\omega = P\,dx + Q\,dy$.

- On D_ε^+, keeping D_ε^+ on the left traverses Σ_R (left-to-right), Γ_{top} (CCW), Σ_L (left-to-right), and $C_{\varepsilon,\text{top}}^-$ (from $(-\varepsilon,0)$ to $(\varepsilon,0)$, clockwise):

$$\iint_{D_\varepsilon^+} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_{\Gamma_{\text{top}}} \omega + \int_{\Sigma_L} \omega + \int_{C_{\varepsilon,\text{top}}^-} \omega + \int_{\Sigma_R} \omega.$$

- On D_ε^-, keeping D_ε^- on the left traverses Γ_{bot} (CCW), Σ_R^- (right-to-left), $C_{\varepsilon,\text{bot}}^-$ (from $(\varepsilon,0)$ to $(-\varepsilon,0)$, clockwise), and Σ_L^- (right-to-left):

$$\iint_{D_\varepsilon^-} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_{\Gamma_{\text{bot}}} \omega + \int_{\Sigma_R^-} \omega + \int_{C_{\varepsilon,\text{bot}}^-} \omega + \int_{\Sigma_L^-} \omega.$$

Step 3 — Addition and cancellation: Using $\int_{\Sigma_L^-}\omega = -\int_{\Sigma_L}\omega$, $\int_{\Sigma_R^-}\omega = -\int_{\Sigma_R}\omega$, and $\int_{C^-_{\varepsilon,\text{top}}}\omega + \int_{C^-_{\varepsilon,\text{bot}}}\omega = -\int_{C^+_{\varepsilon,\text{top}}}\omega - \int_{C^+_{\varepsilon,\text{bot}}}\omega = -\oint_{C^+_\varepsilon}\omega$, adding the two Green–Riemann identities over $D_\varepsilon = D^+_\varepsilon \cup D^-_\varepsilon$ gives

$$\iint_{D_\varepsilon}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy = \left(\int_{\Gamma_{\text{top}}}\omega + \int_{\Gamma_{\text{bot}}}\omega\right) + \underbrace{\left(\int_{\Sigma_L}\omega - \int_{\Sigma_L}\omega\right)}_{=0} + \underbrace{\left(\int_{\Sigma_R}\omega - \int_{\Sigma_R}\omega\right)}_{=0} - \oint_{C^+_\varepsilon}\omega$$

$$= \oint_{\Gamma^+}(P\,dx + Q\,dy) - \oint_{C^+_\varepsilon}(P\,dx + Q\,dy). \qquad \blacksquare$$

> ▸ **Common Pitfall:**
> Writing $+\oint_{C^+_\varepsilon}$ instead of $-\oint_{C^+_\varepsilon}$ confuses counterclockwise orientation in $\mathbb{R}^2$ with positive boundary orientation relative to D_ε: on an *interior* cavity boundary, keeping the domain on the left runs clockwise ($C^-_\varepsilon = -C^+_\varepsilon$).

> ✓ **Sanity Check:**
> Test the identity on $(P, Q) = (0, x)$, where $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$: the right-hand side equals $\iint_{D_\varepsilon} 1\,dx\,dy = \mathcal{A}(D) - \pi\varepsilon^2$, matching $\oint_{\Gamma^+} x\,dy - \oint_{C^+_\varepsilon} x\,dy = \mathcal{A}(D) - \pi\varepsilon^2$ on the left-hand side.

2. Consider $\omega_0 = \dfrac{-y\,dx + x\,dy}{x^2+y^2}$. Explain why the Green–Riemann formula cannot be applied directly to ω_0 on D, then deduce from Question 1 the value of $\oint_{\Gamma^+}\omega_0$. ★★★☆☆

Method *Singularity Obstruction Check and Contour Deformation onto C^+_ε*

> ***Why we do this (Way of Thinking):***
> The Green–Riemann theorem on D requires P_0 and Q_0 to be of class C^1 throughout an open set containing *all* of D, whereas $P_0(x,y) = \frac{-y}{x^2+y^2}$ and $Q_0(x,y) = \frac{x}{x^2+y^2}$ are undefined and unbounded at $(0,0) \in \mathring{D}$. Excising the small disk of radius ε removes $(0,0)$: on D_ε, ω_0 is C^1 and closed ($\frac{\partial Q_0}{\partial x} - \frac{\partial P_0}{\partial y} = 0$), so Question 1 equates $\oint_{\Gamma^+}\omega_0$ over the arbitrary loop Γ^+ to $\oint_{C^+_\varepsilon}\omega_0$ over the circle of radius ε, where $x^2 + y^2 = \varepsilon^2$ is constant.

Execution: Step 1 — Breakdown of Green–Riemann on D: Write $\omega_0 = P_0\,dx + Q_0\,dy$ with $P_0(x,y) = \dfrac{-y}{x^2+y^2}$ and $Q_0(x,y) = \dfrac{x}{x^2+y^2}$. Because $(0,0)$ lies in the interior of D and P_0, Q_0 are undefined and unbounded near $(0,0)$ (for example, $\lim_{x\to 0^+} Q_0(x,0) = \lim_{x\to 0^+}\frac{1}{x} = +\infty$), P_0 and Q_0 do not belong to $C^1(D)$, violating the regularity hypothesis of the Green–Riemann formula on D.

Step 2 — Closedness on $\mathbb{R}^2 \setminus \{(0,0)\}$ and contour transfer via Question 1: On $\mathbb{R}^2 \setminus \{(0,0)\}$, $x^2 + y^2 > 0$, so $P_0, Q_0 \in C^1\big(\mathbb{R}^2 \setminus \{(0,0)\}\big)$, and by the quotient rule:

$$\frac{\partial Q_0}{\partial x}(x,y) = \frac{1\cdot(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2},$$

$$\frac{\partial P_0}{\partial y}(x,y) = \frac{-1\cdot(x^2+y^2) - (-y)(2y)}{(x^2+y^2)^2} = \frac{-x^2 - y^2 + 2y^2}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}.$$

Hence $\dfrac{\partial Q_0}{\partial x} - \dfrac{\partial P_0}{\partial y} = 0$ everywhere on $D_\varepsilon \subset \mathbb{R}^2 \setminus \{(0,0)\}$. Applying Question 1 to (P_0, Q_0) yields

$$\oint_{\Gamma^+}\omega_0 - \oint_{C^+_\varepsilon}\omega_0 = \iint_{D_\varepsilon}\left(\frac{\partial Q_0}{\partial x} - \frac{\partial P_0}{\partial y}\right)dx\,dy = \iint_{D_\varepsilon} 0\,dx\,dy = 0 \quad\Longrightarrow\quad \oint_{\Gamma^+}\omega_0 = \oint_{C^+_\varepsilon}\omega_0.$$

Step 3 — Parameterized evaluation on C_ε^+:

Sub **Parameterize C_ε^+ counterclockwise by $x(t) = \varepsilon\cos t$, $y(t) = \varepsilon\sin t$ for $t \in [0, 2\pi]$, so $dx = -\varepsilon\sin t\,dt$ and $dy = \varepsilon\cos t\,dt$.**

Since $x(t)^2 + y(t)^2 = \varepsilon^2\cos^2 t + \varepsilon^2\sin^2 t = \varepsilon^2$, substituting into $\oint_{C_\varepsilon^+}\omega_0$ gives

$$\oint_{\Gamma^+}\omega_0 = \int_0^{2\pi}\frac{(-\varepsilon\sin t)(-\varepsilon\sin t) + (\varepsilon\cos t)(\varepsilon\cos t)}{\varepsilon^2}\,dt$$
$$= \int_0^{2\pi}\frac{\varepsilon^2(\sin^2 t + \cos^2 t)}{\varepsilon^2}\,dt = \int_0^{2\pi}1\,dt = \Big[t\Big]_{t=0}^{t=2\pi} = 2\pi.$$

$$\boxed{\oint_{\Gamma^+}\omega_0 = 2\pi}$$

> ► **Common Pitfall:**
> Claiming $\omega_0 = d\left(\arctan\frac{y}{x}\right)$ globally and writing $\oint_{\Gamma^+}\omega_0 = 0$ is false because $\arctan\left(\frac{y}{x}\right)$ is undefined on the vertical axis $x = 0$ and increases by 2π around any closed loop encircling the origin.

> ✓ **Sanity Check:**
> The factor ε^2 cancels completely between numerator and denominator in $\oint_{C_\varepsilon^+}\omega_0 = 2\pi$: the value is independent of $\varepsilon > 0$, consistent with the fact that $\oint_{\Gamma^+}\omega_0$ cannot depend on the radius ε of the auxiliary excised disk.

3. Compute

$$K_1 = \oint_{\Gamma_1^+}\left(\frac{x-2y}{x^2+y^2} - y + \sin(x^2)\right)dx + \left(\frac{2x+y}{x^2+y^2} + x + e^{y^2}\right)dy,$$

where Γ_1^+ is the counterclockwise boundary of the rectangle $R_1 = [-1, 3]\times[-1, 2]$. ★★★★☆

Method ***Linearity Decomposition into Exact Radial, Vortex (ω_0), and Smooth (C^1) Forms***

> ***Why we do this (Way of Thinking):***
> On $R_1 = [-1, 3]\times[-1, 2]$, $(0, 0)$ lies strictly inside $\mathring{R}_1$, blocking standard Green–Riemann on the full 1-form, while $\sin(x^2)\,dx$ and $e^{y^2}\,dy$ block direct integration along the four sides of Γ_1^+. Splitting the singular fractions into their radial part $\frac{x\,dx+y\,dy}{x^2+y^2}$ and angular part $2\,\frac{-y\,dx+x\,dy}{x^2+y^2}$ resolves both difficulties simultaneously:
>
> (i) $\frac{x\,dx+y\,dy}{x^2+y^2} = d\left(\frac{1}{2}\ln(x^2+y^2)\right)$ is exact on $\mathbb{R}^2\setminus\{(0, 0)\}$, contributing 0;
>
> (ii) $2\omega_0$ encloses $(0, 0)$ inside R_1, contributing $2(2\pi) = 4\pi$ by Question 2;
>
> (iii) the remaining smooth piece $\left(-y + \sin(x^2)\right)dx + \left(x + e^{y^2}\right)dy$ is C^1 on all of $\mathbb{R}^2$, so standard Green–Riemann on R_1 eliminates $\sin(x^2)$ and e^{y^2} and leaves $\iint_{R_1}2\,dx\,dy$.

Execution: Step 1 — Location of $(0, 0)$ and linear decomposition of the 1-form ω: Since $-1 < 0 < 3$ and $-1 < 0 < 2$, the origin $(0, 0)$ lies strictly in the interior $\mathring{R}_1 =]-1, 3[\times]-1, 2[$, while $\Gamma_1^+ = \partial R_1^+ \subset \mathbb{R}^2\setminus\{(0, 0)\}$ (Figure 5, left). On $\mathbb{R}^2\setminus\{(0, 0)\}$, rewrite $x - 2y = x + 2(-y)$ and $2x + y = y + 2x$ to decompose $\omega = \omega_{\text{rad}} + 2\omega_0 + \omega_{\text{reg}}$, where

$$\omega_{\text{rad}} = \frac{x\,dx + y\,dy}{x^2+y^2}, \qquad \omega_0 = \frac{-y\,dx + x\,dy}{x^2+y^2}, \qquad \omega_{\text{reg}} = \left(-y + \sin(x^2)\right)dx + \left(x + e^{y^2}\right)dy.$$

By linearity of the line integral:

$$K_1 = \oint_{\Gamma_1^+}\omega_{\text{rad}} + 2\oint_{\Gamma_1^+}\omega_0 + \oint_{\Gamma_1^+}\omega_{\text{reg}}.$$

Step 2 — Evaluation of each term:

(i) **Exact radial term:** On $\mathbb{R}^2\setminus\{(0,0)\}$, $h(x,y)=\frac{1}{2}\ln(x^2+y^2)$ belongs to $C^1\big(\mathbb{R}^2\setminus\{(0,0)\}\big)$ with differential

$$dh(x,y)=\frac{1}{2}\cdot\frac{2x}{x^2+y^2}\,dx+\frac{1}{2}\cdot\frac{2y}{x^2+y^2}\,dy=\frac{x\,dx+y\,dy}{x^2+y^2}=\omega_{\mathrm{rad}}.$$

Since $\omega_{\mathrm{rad}}=dh$ is exact on $\mathbb{R}^2\setminus\{(0,0)\}$ and $\Gamma_1^+\subset\mathbb{R}^2\setminus\{(0,0)\}$ is closed, $\displaystyle\oint_{\Gamma_1^+}\omega_{\mathrm{rad}}=0$.

(ii) **Vortex term:** Because $(0,0)$ lies in the interior of R_1 and $\Gamma_1^+=\partial R_1^+$ is a positively oriented piecewise C^1 simple closed curve, Question 2 gives $\displaystyle 2\oint_{\Gamma_1^+}\omega_0=2(2\pi)=4\pi$.

(iii) **Smooth regular term:** Here $P_{\mathrm{reg}}(x,y)=-y+\sin(x^2)$ and $Q_{\mathrm{reg}}(x,y)=x+e^{y^2}$ belong to $C^1(\mathbb{R}^2)$, with

$$\frac{\partial Q_{\mathrm{reg}}}{\partial x}(x,y)-\frac{\partial P_{\mathrm{reg}}}{\partial y}(x,y)=\frac{\partial}{\partial x}\big(x+e^{y^2}\big)-\frac{\partial}{\partial y}\big(-y+\sin(x^2)\big)=1-(-1)=2.$$

Applying the standard Green–Riemann formula on the rectangle $R_1=[-1,3]\times[-1,2]$ of width $3-(-1)=4$, height $2-(-1)=3$, and area $\mathcal{A}(R_1)=4\times3=12$:

$$\oint_{\Gamma_1^+}\omega_{\mathrm{reg}}=\iint_{R_1}2\,dx\,dy=2\,\mathcal{A}(R_1)=2\times12=24.$$

Adding all three contributions yields $K_1=0+4\pi+24=24+4\pi$.

$$\boxed{K_1=24+4\pi}$$

> ► **Common Pitfall:**
> Applying Green–Riemann directly to the unsplit form ω on R_1 computes $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=2$ for $(x,y)\neq(0,0)$ and writes $\iint_{R_1}2\,dx\,dy=24$, missing the singular circulation 4π carried by the puncture at $(0,0)\in\mathring{R}_1$.

> ✓ **Sanity Check:**
> Verify directly via Question 1 on $D_\varepsilon=R_1\setminus B(0,\varepsilon)$ $(0<\varepsilon<1)$: on D_ε, $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=2$, so $\iint_{D_\varepsilon}2\,dx\,dy=2(12-\pi\varepsilon^2)=24-2\pi\varepsilon^2$, while $\oint_{C_\varepsilon^+}\omega=0+4\pi+\iint_{B(0,\varepsilon)}2\,dx\,dy=4\pi+2\pi\varepsilon^2$. Summing gives $(4\pi+2\pi\varepsilon^2)+(24-2\pi\varepsilon^2)=24+4\pi$.

4. Compute the integral of the same differential form over the counterclockwise boundary Γ_2^+ of the rectangle $R_2=[1,3]\times[-1,2]$. ★★☆☆☆

Method *Direct Green–Riemann Formula on a Singularity-Free Rectangle*

> ***Why we do this (Way of Thinking):***
> Unlike $R_1=[-1,3]\times[-1,2]$, every point of $R_2=[1,3]\times[-1,2]$ satisfies $x\geq1>0$, so the singularity $(0,0)$ lies strictly *outside* R_2 (Figure 5, right). Because P and Q are C^1 on the open right half-plane $U_+=\{x>0\}\supset R_2$, standard Green–Riemann applies directly on R_2 with zero singular correction.

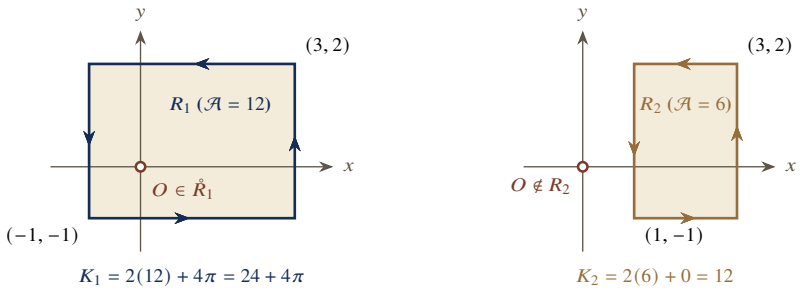


Figure 5: Contrast between $R_1 = [-1,3] \times [-1,2]$ (enclosing the singularity $O(0,0)$, giving $+4\pi$) and $R_2 = [1,3] \times [-1,2]$ (disjoint from $O(0,0)$, giving no singular term).

Execution: Step 1 — Regularity of ω across R_2: Let $K_2 = \oint_{\Gamma_2^+} \omega$. Every $(x,y) \in R_2 = [1,3] \times [-1,2]$ satisfies $x \geq 1 > 0$, so R_2 is contained inside the open right half-plane $U_+ = \{(x,y) \in \mathbb{R}^2 : x > 0\} \subset \mathbb{R}^2 \setminus \{(0,0)\}$. Consequently, $(0,0) \notin R_2$, and the coefficients P and Q of ω belong to $C^1(U_+)$.

Step 2 — Cross-derivative on U_+ and Green–Riemann evaluation: From the decomposition $\omega = \omega_{\text{rad}} + 2\omega_0 + \omega_{\text{reg}}$ in Question 3, both $\omega_{\text{rad}} = dh$ and ω_0 are closed on $U_+ \subset \mathbb{R}^2 \setminus \{(0,0)\}$, whereas $\frac{\partial Q_{\text{reg}}}{\partial x} - \frac{\partial P_{\text{reg}}}{\partial y} = 2$. By linearity, on all of U_+ (and hence on R_2),

$$\frac{\partial Q}{\partial x}(x,y) - \frac{\partial P}{\partial y}(x,y) = 0 + 2(0) + 2 = 2.$$

Applying the Green–Riemann formula directly on the rectangle $R_2 = [1,3] \times [-1,2]$, which has width $3-1 = 2$, height $2-(-1) = 3$, and area $\mathcal{A}(R_2) = 2 \times 3 = 6$:

$$K_2 = \oint_{\Gamma_2^+} \omega = \iint_{R_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy = \iint_{R_2} 2\, dx\, dy = 2\,\mathcal{A}(R_2) = 2 \times 6 = 12.$$

$$\boxed{K_2 = 12}$$

► **Common Pitfall:**
Mechanically adding 4π whenever $2\,\frac{-y\,dx + x\,dy}{x^2+y^2}$ appears in the integrand—without checking whether $(0,0)$ lies inside or outside the contour—would falsely give $12 + 4\pi$. When $(0,0) \notin R_2$, $\oint_{\Gamma_2^+} \omega_0 = \iint_{R_2} 0\, dx\, dy = 0$.

✓ **Sanity Check:**
Partition $R_1 = [-1,3] \times [-1,2]$ along the vertical segment $x = 1$ into $R_{\text{left}} = [-1,1] \times [-1,2]$ (area 6, containing $(0,0)$) and $R_2 = [1,3] \times [-1,2]$ (area 6, excluding $(0,0)$). By cancellation along the shared edge $x = 1$, $K_1 = \oint_{\partial R_{\text{left}}^+} \omega + K_2 = (2 \times 6 + 4\pi) + 12 = 24 + 4\pi$, matching Question 3 identically.