



Work-Sheet 01: Real Numbers

Analysis 1 — 2026/2027

Methods of Proof · Absolute Values · Bounded Sets · Floor Function

Part I: Methods of Proof

1) Direct proof

Exercise 01:

Let $a, b, c \in \mathbb{R}$.

- 1) Prove that $a^2 + b^2 + c^2 \geq ab + bc + ac$.
- 2) Show that the equality holds only if $a = b = c$.
- 3) Deduce that if $a + b + c = 3$, then $ab + ac + bc \leq 3$.

Exercise 02:

Let $a, b, c > 0$. Prove that:

- 1) $(a + b)(b + c)(c + a) \geq 8abc$.
- 2) $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3$.

2) Proof by cases

Exercise 01:

- 1) Show that $\forall n \in \mathbb{N} : n^2 + 3n + 1$ is odd.
- 2) Show that $\forall (x, y) \in \mathbb{R}^2, \max(x, y) + \min(x, y) = x + y$.

3) Induction

Exercise 01:

Prove that:

- 1) $\forall n \geq 4 : n^2 \leq 2^n$.
- 2) $\sum_{k=1}^n \frac{1}{k^2} \leq 2 - \frac{1}{n}$.
- 3) $\forall n \in \mathbb{N} \setminus \{0\} : n! \geq 2^{n-1}$.

Exercise 02:

Show by induction that, for every $n \in \mathbb{N}^*$,

$$\underbrace{\sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}}_{n\text{-times}} = 2 \cos\left(\frac{\pi}{2^{n+1}}\right).$$

Exercise 03:

Let $x_1, \dots, x_n \geq -1$ and all of the same sign.

1) Show that

$$(1 + x_1) \cdots (1 + x_n) \geq 1 + x_1 + \cdots + x_n.$$

2) Deduce that $(1 + x)^n \geq 1 + nx$.

4) Counterexample

Exercise 01:

Disprove the following statements:

a) $n^2 + n + 41$ is a prime number, $n \in \mathbb{N}$.

b) $\forall (a, b) \in \mathbb{R}^2$, if $a^2 \leq b^2$ then $a \geq b$.

5) Contradiction

Exercise 01:

Let a, b, c be odd integers. Show that

$$ax^2 + bx + c = 0$$

has no rational solution.

Part II: Absolute Values

Exercise 01:

1) Show that $\forall (a, b) \in \mathbb{R}^2$,

$$\max(a, b) = \frac{a + b + |a - b|}{2} \quad \text{and} \quad \min(a, b) = \frac{a + b - |a - b|}{2}.$$

2) Deduce that $\forall a, b, c, d \in \mathbb{R}$,

$$|\max(a, b) - \max(c, d)| \leq \max(|a - c|, |b - d|).$$

Exercise 02:

Solve in $\mathbb{R}$:

a) $|x - 1| + |x - 2| + |x - 3| = 2$

b) $|x^2 - 2x + 3| \leq 2|x + 1|$

c) $||x - 1| - 2| > |x| - 1$

Exercise 03:

Let $x, y \in \mathbb{R}$ with $|x| \leq 1$ and $|y| \leq 1$. Prove that:

1) $|x + y| \leq 1 + xy$.

2) $\frac{|x - y|}{|1 - xy|} \leq 1$ when $xy \neq 1$.

Part III: Bounded Sets

$$A_1 = \left\{ \forall n \in \mathbb{N} : \frac{n^2}{2^n} \right\}$$

$$A_2 = \left\{ \forall n \in \mathbb{N}^* : \sqrt[n]{1 + (-1)^n} \right\}$$

$$A_3 = \left\{ \left(1 - \frac{1}{n}\right) \cos\left(\frac{n\pi}{2}\right), n \in \mathbb{N}^* \right\}$$

$$A_4 = \{x \in \mathbb{R} : \lfloor x \rfloor^3 - 3 \lfloor x \rfloor + 2 = 0\}$$

$$A_5 = \left\{ \forall n \in \mathbb{N}^* : (-1)^n \left(4 - \frac{3}{n}\right) \right\}$$

$$A_6 = \left\{ \forall n \in \mathbb{N}^* : 2 \cos\left(\frac{n(n+1)}{4}\pi\right) \left(1 - \frac{1}{n}\right) \right\}$$

Part IV: Floor Function

Exercise 01:

1) Solve in $\mathbb{R}$ the following equation:

$$(E) : \frac{\lfloor x \rfloor}{\lfloor x - 2 \rfloor} - \frac{\lfloor x - 2 \rfloor}{\lfloor x \rfloor} = \frac{8 \lfloor x \rfloor + 12}{\lfloor x \rfloor \lfloor x - 2 \rfloor}.$$

2) Let A be the set of the solutions of (E) .

2.1) Determine $\sup(A)$, $\inf(A)$, $\max(A)$, $\min(A)$.

2.2) Use a complete proof to show that $\sup(A) = -3$.

Exercise 02:

Let $n \in \mathbb{N}^*$ and $x \in \mathbb{R}$. Prove that:

a) $\left\lfloor \frac{\lfloor x \rfloor}{n} \right\rfloor = \left\lfloor \frac{x}{n} \right\rfloor$

b) $\lfloor nx \rfloor = \sum_{k=0}^{n-1} \left\lfloor x + \frac{k}{n} \right\rfloor$

c) $\forall x \geq 0 : \left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor = \lfloor \sqrt{x} \rfloor$

Exercise 03:

Solve in $\mathbb{R}$:

a) $x^2 - 5 \lfloor x \rfloor + 6 = 0$

b) $\lfloor \sqrt{x^2 + 1} \rfloor = 3$

c) $\lfloor 2x - 1 \rfloor < \lfloor x \rfloor + 1$

d) $\left\lfloor \frac{x}{2} \right\rfloor \geq \frac{x-1}{3}$

Exercise 04:

Let $y \in \mathbb{R}$.

1) Prove that $\lfloor y \rfloor + \lfloor y + \frac{1}{2} \rfloor = \lfloor 2y \rfloor$.

2) Show that

$$n = \sum_{k=0}^m \left\lfloor \frac{n + 2^k}{2^{k+1}} \right\rfloor.$$