Solutions – TD N°02 : Conductors in Static Equilibrium

Exercise 1 – Spherical Capacitor

Given: R = 1 cm = 0.01 m, R_1 = 3 cm = 0.03 m, R_2 = 5 cm = 0.05 m, EMF = 24 V, $k = 1/(4\pi\varepsilon_0) = 9\times10^9$ $N\cdot m^2/C^2$

Question 1 – Charge distribution on the inner sphere

The inner sphere is a solid metal conductor connected to a positive DC supply. In electrostatic equilibrium, all free charges reside on the **outer surface** of the inner sphere (at r = R). There are **no free charges inside** a conductor. The charge is uniformly distributed over the spherical surface (by symmetry).

Question 2 – Charge inside the inner sphere (r < R)

Inside a conductor in electrostatic equilibrium the electric field is zero. By Gauss's law, the enclosed free charge must also be zero:

$$Q_{inside} = 0$$

Question 3 – Charge on the inner sphere surface (r = R)

The inner sphere forms a spherical capacitor with the inner surface of the shell. The capacitance of a spherical capacitor is:

$$C = 4\pi\varepsilon_0 \cdot (R \cdot R_1) / (R_1 - R)$$

Numerical application:

$$C = (0.01 \times 0.03) / [9\times10^9 \times (0.03 - 0.01)] = (3\times10^{-4}) / (1.8\times10^8)$$

$$C = 1.6667 \text{ pF} \approx 1.667 \text{ pF}$$

$$Q = C \times V = 1.6667\times10^{-12} \times 24 = 0.0400 \text{ nC}$$

Q (on inner sphere surface at r = R) = +0.0400 nC ≈ 40.00 pC

Question 4 – Charge on the inner surface of the shell (r = R_1)

By Gauss's law applied to a Gaussian surface drawn inside the conducting shell ($R_1 < r < R_2$): the electric field is zero there, so the total enclosed charge is zero. Therefore the inner surface of the shell must carry a charge equal and opposite to that of the inner sphere:

$Q_{inner\ shell}$ = –Q = –0.0400 nC

Question 5 – Charge on the outer surface of the shell (r = R_2)

The outer shell was initially neutral (total charge = 0). The inner surface acquired –Q, so by conservation of charge the outer surface must carry +Q:

$Q_{outer\ shell}$ = +Q = +0.0400 nC

Note: The outer conductor is NOT grounded in this problem, so charge can appear on its outer surface.

Question 6 – Surface charge densities

The surface charge density is $\sigma = Q / (4\pi r^2)$.

Surface	Radius	Charge	σ = Q / 4πr²
Inner sphere	R = 1 cm	+0.0400 nC	0.0318 μC/m²
Inner surface shell	R■ = 3 cm	–0.0400 nC	-0.0035 μC/m²
Outer surface shell	R■ = 5 cm	+0.0400 nC	0.0013 μC/m²

Conclusion: The charge densities are **inversely proportional to r²**. The inner sphere (smallest radius) has the highest surface charge density; the outer shell surface (largest radius) has the lowest.

Question 7 – Electric field E(r) via Gauss's Law

We apply Gauss's law: ■ **E**·d**A** = Q_{enc}/ε_0 with a spherical Gaussian surface of radius r and area 4πr².

Region 1 – Inside the inner sphere (r < R)

The inner sphere is a conductor → E = 0.

```
E(r) = 0
```

No field lines inside the conductor.

Region 2 – Between sphere and shell ($R < r < R_1$)

Enclosed charge = +Q (from the inner sphere surface).

```
E(r) = Q / (4πε■ r²) = kQ / r² [radially outward]
```

At r = R: E = 3600.0 V/m At r = R■: E = 400.0 V/m

Region 3 – Inside the conducting shell ($R_1 < r < R_2$)

Inside a conductor → E = 0.

```
E(r) = 0
```

No field lines inside the metal shell.

Region 4 – Outside the shell ($r > R_2$)

Enclosed charge = +Q – Q + Q = +Q (net charge of the system).

```
E(r) = kQ / r² [radially outward]
```

Same form as Region 2 but for r > R■.

Exercise 2 – Cylindrical Capacitor

Given: r_1 = 1 mm = 10^{-3} m, r_2 = 3 mm = 3×10^{-3} m, r_3 = 4 mm = 4×10^{-3} m, L = 10 cm = 0.1 m, ε = 12 V

Question 1 – Charge distribution

Inner conductor A is connected to ε = 12 V and outer conductor B is grounded (V = 0).

- **On the surface of inner cylinder A ($r = r_1$):** All free charges reside on the outer surface of A. Let the total charge be +Q (positive, since A is at higher potential).
- **Region $r_2 < r < r_3$ (inside conductor B):** E = 0 inside a conductor. By Gauss's law the total enclosed charge is zero, so the inner surface of B (at $r = r_2$) carries –Q.
- **Outer surface of B ($r = r_3$):** Conductor B is grounded, so any excess charge flows to ground. The outer surface carries **zero charge** (grounding ensures $V_{outer} = 0$ and $Q_{outer} = 0$).

Question 2 – Electric field E(r)

By Gauss's law with a coaxial cylindrical surface of radius r and length ■: E × 2πr■ = Q_{enc} / ε_0 → E = λ / ($2\pi\varepsilon_0$ r), where λ = Q/L is the linear charge density.

Region	Description	E(r)
$r < r_1$	Inside conductor A	`E = 0`
$r_1 < r < r_2$	Between A and B	`E(r) = λ / (2πε■ r) = Q / (2πε■ L r) [radially outward]`
$r_2 < r < r_3$	Inside conductor B	`E = 0`
$r > r_3$	Outside (B is grounded, net charge = 0)	`E = 0`

Question 3 – Potential difference between A and inner surface of B

Integrating E(r) from r_1 to r_2:

```
V(r■) − V(r■) = ∫[r■ to r■] E dr = (λ / 2πε■) × ln(r■/r■)
```

Since the outer conductor B is grounded, V(r■) = 0, therefore:

```
V(r■) − V(r■) = ε = 12 V (the applied emf)
```

$\Delta V = V_A - V_B = 12$ V

Question 4 – Curves of E(r) and V(r)

The key features to sketch:

- **E(r):** Zero for $r < r_1$ and $r > r_2$ (conductors + grounded outer). For $r_1 < r < r_2$: E ∝ 1/r, ranging from $E(r_1)$ = 10922.9 V/m down to $E(r_2)$ = 3641.0 V/m.
- **V(r):** V = ε = 12 V for $r \le r_1$ (inside/on A). Decreases logarithmically as V(r) = $(\lambda/2\pi\varepsilon_0) \ln(r_2/r)$ for $r_1 \le r \le r_2$. V = 0 for $r \ge r_2$ (grounded B).

Question 5 – Capacitance C

From Q = C × ΔV and ΔV = $(\lambda/2\pi\varepsilon_0) \ln(r_2/r_1)$:

```
C = Q / ΔV = 2πε■ L / ln(r■/r■)
```

Numerical application:

$$C = 2\pi \times 8.854\times10^{-12} \times 0.1 / \ln(3\times10^{-3} / 10^{-3})$$

$$C = 2\pi \times 8.854\times10^{-12} \times 0.1 / \ln(3) = 5.0638 \text{ pF}$$

C = 5.0638 pF ≈ 5.06 pF

Question 6 – Electrostatic energy stored

The energy stored in a capacitor is:

$$U = ½ C V^2 = ½ \times C \times \varepsilon^2$$

$$U = ½ \times 5.0638\times10^{-12} \times (12)^2$$

U = 364.5922 pJ ≈ 364.59 pJ

This can also be verified by integrating the energy density u = $½\varepsilon_0 E^2$ over the volume between the cylinders:

```
U = ∫ ½ε■ E² dV = (λ² L / 4πε■) × ln(r■/r■) [which gives the same result]
```

Question 7 – Approximate expression when $r_2 = r_1 + e$, $e << r_1$

When $e = r_2 - r_1 << r_1$, use the approximation $\ln(1 + x) \approx x$ for $x << 1$:

ln(r■/r■) = ln(1 + e/r■) ≈ e/r■

Therefore:

C ≈ 2πε■ L r■ / e

Question 8 – Agreement with plane capacitor formula

For a plane (parallel-plate) capacitor of plate area A, plate separation d:

C_{plane} = ε■ A / d

When the gap e is very small compared to r_1, the curved surface of the inner cylinder is approximately flat. The relevant area is the lateral surface of the inner cylinder:

A = 2π r■ L (lateral surface of inner cylinder)

Substituting A and d = e into the plane capacitor formula:

C_{plane} = ε■ × (2π r■ L) / e = 2πε■ L r■ / e

This is identical to the approximate cylindrical result from Question 7. ✓ **Consistent.**
