Revision Set

Exercise 1:

1) Let $\alpha \in \mathbb{R}$. In each of the following cases, determine, depending on the value of α, a basis for $\ker(f_\alpha)$ and a basis for $\operatorname{Im}(f_\alpha)$.

   (a) $f_\alpha : \begin{array}{ccc} \mathbb{R}^4 & \longrightarrow & \mathbb{R}^3 \\ (x,y,z,t) & \longmapsto & (2x+y+(\alpha+1)z+2t, x-y+t, x+(1-\alpha)y+t) \end{array}$

   (b) $f_\alpha : \begin{array}{ccc} \mathbb{R}_2[X] & \longrightarrow & \mathbb{R}^3 \\ P & \longmapsto & (P(1)-\alpha P(0), \alpha P'(-1), \alpha^2 P''(0)) \end{array}$

   (c) $f_\alpha : \begin{array}{ccc} \mathbb{R}_3[X] & \longrightarrow & \mathbb{R}_3[X] \\ P & \longmapsto & \alpha X P' + P(\alpha) \end{array}$

2) Let $n, m \in \mathbb{N}$. Depending on the values of the parameters n and m, determine the kernel of the linear map:

$$g : \begin{array}{ccc} \mathbb{R}_n[X] & \longrightarrow & \mathbb{R}^m \\ P & \longmapsto & (P(1), P'(1), \ldots, P^{(m)}(1)) \end{array}$$

Exercise 2:

Let f be the endomorphism of $\mathbb{R}^3$ defined by $f(x,y,z) = (y, z, x-3y+3z)$. Let $v_1 := (1,1,1)$, $v_2 := (1,2,3)$, and $v_3 := (-1,0,2)$.

1) Show that the set $\{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$.

2) Express $f(av_1 + bv_2 + cv_3)$ as a linear combination of v_1, v_2, and v_3, where $a, b, c \in \mathbb{R}$.

3) Deduce $\ker f$, and show that f is bijective.

4) Compute $(f - \mathrm{Id}_{\mathbb{R}^3})^3(v_i)$ for $i = 1,2,3$ and deduce a formula for f^{-1} as a polynomial in f.

Exercise 3:

1) Are there linear maps from $\mathbb{R}^4$ into $\mathbb{R}^2$ whose kernel is $H := \{(x,y,z,t) \in \mathbb{R}^4 : x = y = 2z = 3t\}$?

2) Is there an endomorphism of $\mathbb{R}^3$ whose kernel is equal to its image?

3) Show that there exists a unique endomorphism f of $\mathbb{R}^4$ such that $\ker(f) = \{(x,y,z,t) \in \mathbb{R}^4, y-z-t = x+3z+2t = 0\}$, $f(1,0,0,0) = (1,-1,1,0)$ and $f(2,0,0,3) = (0,1,0,0)$.

Exercise 4:

Let V and W be two K-vector spaces, $f \in \mathcal{L}(V, W)$, and X, Y be two subspaces of V.

1) Show that $f(X + Y) = f(X) + f(Y)$.
2) Show that if the sum $X + Y$ is direct and f is injective, then $f(X \oplus Y) = f(X) \oplus f(Y)$.

Exercise 5:

1) Let $a, b \in \mathbb{Z}$. Consider the matrices:

$$A = \begin{pmatrix} a & 2 \\ b & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$$

   Given that $\det(A^T B) = 10$, use the properties of determinants and divisors to:

   (a) Find all possible integer pairs (a, b).

2) The 3×3 matrix M is defined in terms of the scalar constant k by:

$$M = \begin{pmatrix} 2 & k & 1 \\ 3 & -2 & 4 \\ 2k & 3 & 7 \end{pmatrix}$$

   Given that $\det(M) = 8$, find the possible values of k.