Revision Set 2

Exercise 1:

Let $\alpha, \beta \in \mathbb{R}$. Determine the values of α and β for which the linear map associated with the following matrix is surjective:

$$M_{\alpha,\beta} = \begin{pmatrix} 1 & 3 & \alpha & \beta \\ 2 & -1 & 2 & 1 \\ -1 & 1 & 2 & 0 \end{pmatrix}$$

Exercise 2:

Let $E = \mathbb{R}_n[X]$ be the vector space of polynomials of degree at most n. Consider the map u defined by:

$$u(P) = (X^2 - 1)P'' + (2X + 1)P'$$

1. Verify that u is an endomorphism of E.
2. Write the matrix of u in the standard basis of $\mathbb{R}_2[X]$ (case $n = 2$).
3. Determine the kernel and the image of u. Is the map bijective?

Exercise 3:

Let $A = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}$ and let f be the application from $\mathcal{M}_2(\mathbb{R})$ to $\mathcal{M}_2(\mathbb{R})$ defined for all $M \in \mathcal{M}_2(\mathbb{R})$ by $f(M) = AM$.

1. Show that f is a linear application.
2. Determine the matrix of f relative to the canonical basis $\mathcal{B} = (E_1, E_2, E_3, E_4)$ of $\mathcal{M}_2(\mathbb{R})$.

Exercise 4:

Let $V = \mathbb{R}_2[x]$. Consider $f : V \to V$ defined by:

$$f(P)(x) = xP'(x) - \int_0^1 P(t)\,dt.$$

Let $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{B}' = \{1 + x,\ x + x^2,\ 1 + x^2\}$.

(a) Compute $A = M(f, \mathcal{B})$.

(b) Construct $P = P_{\mathcal{B} \to \mathcal{B}'}$ and compute P^{-1}.

(c) For $Q(x) = 2x^2 - 1$, verify that $[Q]_{\mathcal{B}'} = P^{-1}[Q]_{\mathcal{B}}$.

Exercise 5:

Let $u : \mathbb{R}^3 \to \mathbb{R}^2$ have matrix $A = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 2 & -3 \end{pmatrix}$ relative to canonical bases. Define:

$$e_1' = e_2 + e_3, \quad e_2' = e_3 + e_1, \quad e_3' = e_1 + e_2 \quad \text{and} \quad f_1' = \tfrac{1}{2}(f_1 + f_2), \quad f_2' = \tfrac{1}{2}(f_1 - f_2).$$

1. Show that $\mathcal{B}' = (e_1', e_2', e_3')$ and $\mathcal{C}' = (f_1', f_2')$ are bases.
2. Determine the matrix of u in these new bases.

Exercise 6:

Consider the space $E = \mathbb{R}_3[X]$ and the map $f(P) = P(X+1) - P(X)$.

1. Represent f by a matrix M in the standard basis.
2. Determine the rank of M. Does the equation $f(P) = Q$ always have a solution for any $Q \in \mathbb{R}_2[X]$?

Exercise 7:

Let $\mathcal{M}_n(\mathbb{R})$, $a \in \mathbb{R}$, and A, B invertible matrices. Solve for X:

(a) $aX + \mathrm{tr}(X)A = B$

(b) $aX + \mathrm{tr}(XA)I_n = B$

(c) $\det(A)\, XB + \mathrm{tr}(XA)\, B = A$

Exercise 8:

Find values of a and m such that the system of linear equations

$$\begin{cases} x - y + 2z = a \\ mx + (1-m)y + (2m-2)z = m \\ 2x + my - (3m+1)z = 2a \end{cases}$$

has (1) exactly one solution, (2) infinitely many solutions, (3) no solution.

Exercise 9:

Let m be a real number. Solve the following system:

$$\begin{cases} x + my = -3 \\ mx + 4y = 6 \end{cases}$$

(the discussion may be conducted in terms of m). What geometric interpretation do you make of the result?

Exercise 10:

Balance the following chemical equations:

1. $NH_3 + O_2 \to NO + H_2O$;
2. $C_2H_6 + O_2 \to CO_2 + H_2O$.