Solutions to Revision Set 2

Exercise 1

Goal: A linear map is surjective if and only if its matrix has maximal row rank, i.e. $\operatorname{rank}(M_{\alpha,\beta}) = 3$. We reduce the matrix by row operations and look at when the third pivot can become zero.

Step 1 — Eliminate the first column below the pivot.

$$\begin{pmatrix} 1 & 3 & \alpha & \beta \\ 2 & -1 & 2 & 1 \\ -1 & 1 & 2 & 0 \end{pmatrix} \xrightarrow[L_3 \leftarrow L_3 + L_1]{L_2 \leftarrow L_2 - 2L_1} \begin{pmatrix} 1 & 3 & \alpha & \beta \\ 0 & -7 & 2-2\alpha & 1-2\beta \\ 0 & 4 & 2+\alpha & \beta \end{pmatrix}$$

Step 2 — Eliminate the second column in L_3. We do $L_3 \leftarrow 7L_3 + 4L_2$ to avoid fractions:

$$\xrightarrow{L_3 \leftarrow 7L_3 + 4L_2} \begin{pmatrix} 1 & 3 & \alpha & \beta \\ 0 & -7 & 2-2\alpha & 1-2\beta \\ 0 & 0 & 22-\alpha & 4-\beta \end{pmatrix}$$

Step 3 — Read the rank. The matrix has rank 3 if and only if the third row is not entirely zero, i.e. not both $22-\alpha = 0$ and $4-\beta = 0$ simultaneously.

Conclusion: The map is surjective for all $(\alpha, \beta) \in \mathbb{R}^2 \setminus \{(22, 4)\}$.

> ▶ Note
>
> The rank drops to 2 only when **both** conditions fail at the same time ($\alpha = 22$ **and** $\beta = 4$). If only one of them holds, the rank is still 3 and the map is still surjective.

Exercise 2

We work in $E = \mathbb{R}_2[X]$ with standard basis $\mathcal{B} = (1, X, X^2)$. The map is $u(P) = (X^2-1)P'' + (2X+1)P'$.

1. Showing u is an endomorphism of E.

- **Linearity:** For any $P, Q \in E$ and $\lambda \in \mathbb{R}$:

  $$u(P + \lambda Q) = (X^2-1)(P+\lambda Q)'' + (2X+1)(P+\lambda Q)' = u(P) + \lambda\, u(Q).$$

  This follows directly from the linearity of differentiation.

- **Stability:** If $\deg(P) \le 2$, then $\deg(P') \le 1$ and $\deg(P'') \le 0$. Therefore:

  $$\deg\big((X^2-1)P''\big) \le 2 + 0 = 2, \qquad \deg\big((2X+1)P'\big) \le 1 + 1 = 2.$$

  Hence $\deg(u(P)) \le 2$, so $u(P) \in \mathbb{R}_2[X]$. The space E is stable.

2. Matrix of u in $\mathcal{B} = (1, X, X^2)$.

We apply u to each basis vector. Each image is then expressed as a coordinate column in $\mathcal{B}$.

$u(1)$: $P = 1$, so $P' = 0$ and $P'' = 0$.

$$u(1) = (X^2 - 1) \cdot 0 + (2X + 1) \cdot 0 = 0 \implies [u(1)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$u(X)$: $P = X$, so $P' = 1$ and $P'' = 0$.

$$u(X) = (X^2 - 1) \cdot 0 + (2X + 1) \cdot 1 = 1 + 2X \implies [u(X)]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

$u(X^2)$: $P = X^2$, so $P' = 2X$ and $P'' = 2$.

$$u(X^2) = (X^2 - 1) \cdot 2 + (2X + 1) \cdot 2X = 2X^2 - 2 + 4X^2 + 2X = 6X^2 + 2X - 2 \implies [u(X^2)]_{\mathcal{B}} = \begin{pmatrix} -2 \\ 2 \\ 6 \end{pmatrix}.$$

Placing these as columns:

$$M_{\mathcal{B}}(u) = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 2 & 2 \\ 0 & 0 & 6 \end{pmatrix}.$$

3. Kernel and image.

The matrix is upper-triangular with diagonal $(0, 2, 6)$. The only zero diagonal entry is in position $(1, 1)$, meaning the first column is the zero vector, which corresponds to $u(1) = 0$.

- **Kernel:** $\ker(u) = \operatorname{span}(1)$, with dimension 1.
- **Image:** By the rank–nullity theorem, $\dim(\operatorname{Im}(u)) = 3 - 1 = 2$. The non-zero images give a basis: $\operatorname{Im}(u) = \operatorname{span}(1 + 2X,\ -2 + 2X + 6X^2)$.
- **Not bijective:** The kernel is non-trivial, so u is neither injective nor surjective (as an endomorphism).

▶ Note

When computing $u(X^2)$, be careful with the product rule: $(X^2 - 1) \cdot 2$ gives the -2 constant term, not zero. A common mistake is to forget that the coefficient $(X^2 - 1)$ evaluates to a non-zero constant.

Exercise 3

Let $A = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}$ and $f(M) = AM$. The canonical basis of $\mathcal{M}_2(\mathbb{R})$ is:

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

1. Linearity. For any $M, N \in \mathcal{M}_2(\mathbb{R})$ and $\lambda \in \mathbb{R}$:

$$f(M + \lambda N) = A(M + \lambda N) = AM + \lambda AN = f(M) + \lambda f(N).$$

Matrix multiplication distributes over addition and scalar multiplication, so f is linear.

2. Matrix of f. Each column of $[f]_{\mathcal{B}}$ is the image of the corresponding basis matrix, written as a coordinate vector in the basis (E_1, E_2, E_3, E_4). We compute AE_j for each j:

$f(E_1) = AE_1 = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} = -E_1 + E_3 \implies$ column $= (-1, 0, 1, 0)^T$.

$f(E_2) = AE_2 = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} = -E_2 + E_4 \implies$ column $= (0, -1, 0, 1)^T$.

$f(E_3) = AE_3 = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = 2E_1 \implies$ column $= (2, 0, 0, 0)^T$.

$f(E_4) = AE_4 = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 2E_2 \implies$ column $= (0, 2, 0, 0)^T$.

Assembling the columns:

$$[f]_{\mathcal{B}} = \begin{pmatrix} -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

> **▶ Note**
>
> $f(M) = AM$ is *left*-multiplication by A. The column of index j in the matrix of f is the coordinate vector of $A \cdot E_j$, not $E_j \cdot A$. These differ whenever A is not symmetric.

Exercise 4

We have $f(P)(x) = xP'(x) - \int_0^1 P(t)\,dt$, with $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{B}' = \{1 + x,\ x + x^2,\ 1 + x^2\}$.

(a) Computing $A = M(f, \mathcal{B})$.

We apply f to each basis vector. The integral $\int_0^1 P(t)\,dt$ is a *constant*, so it only affects the constant term of $f(P)$.

$f(1)$: $P = 1$, $P' = 0$, $\int_0^1 1\,dt = 1$.

$$f(1)(x) = x \cdot 0 - 1 = -1 \implies [f(1)]_{\mathcal{B}} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}.$$

$f(x)$: $P = x$, $P' = 1$, $\int_0^1 t\,dt = \frac{1}{2}$.

$$f(x)(x) = x \cdot 1 - \tfrac{1}{2} = -\tfrac{1}{2} + x \implies [f(x)]_{\mathcal{B}} = \begin{pmatrix} -1/2 \\ 1 \\ 0 \end{pmatrix}.$$

$f(x^2)$: $P = x^2$, $P' = 2x$, $\int_0^1 t^2\,dt = \frac{1}{3}$.

$$f(x^2)(x) = x \cdot 2x - \tfrac{1}{3} = -\tfrac{1}{3} + 2x^2 \implies [f(x^2)]_{\mathcal{B}} = \begin{pmatrix} -1/3 \\ 0 \\ 2 \end{pmatrix}.$$

$$A = M(f, \mathcal{B}) = \begin{pmatrix} -1 & -\frac{1}{2} & -\frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

(b) Change-of-basis matrix $P = P_{\mathcal{B}\to\mathcal{B}'}$ and its inverse.
Each column of P contains the coordinates of a $\mathcal{B}'$ vector expressed in $\mathcal{B}$:

$$1 + x = 1 \cdot 1 + 1 \cdot x + 0 \cdot x^2, \quad x + x^2 = 0 \cdot 1 + 1 \cdot x + 1 \cdot x^2, \quad 1 + x^2 = 1 \cdot 1 + 0 \cdot x + 1 \cdot x^2.$$

$$P = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

To find P^{-1}, we compute $\det(P)$ by cofactor expansion along the first row:

$$\det(P) = 1 \cdot (1 \cdot 1 - 0 \cdot 1) - 0 + 1 \cdot (1 \cdot 1 - 1 \cdot 0) = 1 + 1 = 2.$$

Since $\det(P) \neq 0$, $\mathcal{B}'$ is indeed a basis. The adjugate (transpose of cofactors) gives:

$$P^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}.$$

Verification: $PP^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = I_3$. ✓

(c) Verifying $[Q]_{\mathcal{B}'} = P^{-1}[Q]_{\mathcal{B}}$ for $Q(x) = 2x^2 - 1$.
First, the coordinates of Q in $\mathcal{B}$:

$$Q = -1 \cdot 1 + 0 \cdot x + 2 \cdot x^2 \implies [Q]_{\mathcal{B}} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}.$$

Now apply P^{-1}:

$$P^{-1}[Q]_{\mathcal{B}} = \frac{1}{2}\begin{pmatrix}1 & 1 & -1\\ -1 & 1 & 1\\ 1 & -1 & 1\end{pmatrix}\begin{pmatrix}-1\\ 0\\ 2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-1+0-2\\ 1+0+2\\ -1+0+2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-3\\ 3\\ 1\end{pmatrix} = \begin{pmatrix}-3/2\\ 3/2\\ 1/2\end{pmatrix}.$$

Verification: Reconstruct Q from these coordinates in $\mathcal{B}'$:

$$-\tfrac{3}{2}(1+x)+\tfrac{3}{2}(x+x^2)+\tfrac{1}{2}(1+x^2) = \left(-\tfrac{3}{2}+\tfrac{1}{2}\right)+\left(-\tfrac{3}{2}+\tfrac{3}{2}\right)x+\left(\tfrac{3}{2}+\tfrac{1}{2}\right)x^2 = -1+2x^2 = Q(x).\checkmark$$

▶ Note

The change-of-basis matrix $P = P_{\mathcal{B}\to\mathcal{B}'}$ has the $\mathcal{B}'$ vectors as columns, expressed in $\mathcal{B}$. A common error is to build it the other way around. Always check: $P\cdot[v]_{\mathcal{B}'} = [v]_{\mathcal{B}}$.

Exercise 5

Let $u : \mathbb{R}^3 \to \mathbb{R}^2$ with matrix $A = \begin{pmatrix}2 & -1 & 1\\ 3 & 2 & -3\end{pmatrix}$ in canonical bases. New bases are defined by:

$$e_1' = e_2+e_3,\quad e_2' = e_3+e_1,\quad e_3' = e_1+e_2,\qquad f_1' = \tfrac{1}{2}(f_1+f_2),\quad f_2' = \tfrac{1}{2}(f_1-f_2).$$

1. Showing $\mathcal{B}' = (e_1', e_2', e_3')$ and $\mathcal{C}' = (f_1', f_2')$ are bases.

Write the change-of-basis matrices by placing the new vectors as columns (in canonical coordinates):

$$P = \begin{pmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix},\qquad Q = \begin{pmatrix}1/2 & 1/2\\ 1/2 & -1/2\end{pmatrix}.$$

Compute the determinants:

$$\det(P) = 0(0-1) - 1(0-1) + 1(1-0) = 0+1+1 = 2 \neq 0,$$
$$\det(Q) = \tfrac{1}{2}\cdot\left(-\tfrac{1}{2}\right) - \tfrac{1}{2}\cdot\tfrac{1}{2} = -\tfrac{1}{4}-\tfrac{1}{4} = -\tfrac{1}{2} \neq 0.$$

Since both determinants are non-zero, $\mathcal{B}'$ and $\mathcal{C}'$ are indeed bases of $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively.

2. Matrix of u in the new bases.

The change-of-basis formula for a map between *two different* spaces is:

$$A' = Q^{-1}AP$$

where P changes from the new domain basis to the old one, and Q^{-1} changes from the old codomain basis to the new one.

Compute Q^{-1}: Since $\det(Q) = -\tfrac{1}{2}$,

$$Q^{-1} = \frac{1}{-1/2}\begin{pmatrix}-1/2 & -1/2\\ -1/2 & 1/2\end{pmatrix} = \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}.$$

Compute AP first:

$$AP = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 2 & -3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0-1+1 & 2+0+1 & 2-1+0 \\ 0+2-3 & 3+0-3 & 3+2+0 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 1 \\ -1 & 0 & 5 \end{pmatrix}.$$

Then compute $Q^{-1}(AP)$:

$$A' = Q^{-1}AP = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 3 & 1 \\ -1 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 0-1 & 3+0 & 1+5 \\ 0+1 & 3-0 & 1-5 \end{pmatrix} = \begin{pmatrix} -1 & 3 & 6 \\ 1 & 3 & -4 \end{pmatrix}.$$

The matrix of u in bases $\mathcal{B}'$ and $\mathcal{C}'$ is $A' = \begin{pmatrix} -1 & 3 & 6 \\ 1 & 3 & -4 \end{pmatrix}$.

▶ Note

For a map $u : E \to F$ between *different* spaces, the formula is $A' = Q^{-1}AP$ (two *different* matrices P and Q). The formula $A' = P^{-1}AP$ only applies to endomorphisms ($E = F$, same basis on both sides).

Exercise 6

We work in $E = \mathbb{R}_3[X]$ with standard basis $\mathcal{B} = (1, X, X^2, X^3)$. The map is $f(P) = P(X+1) - P(X)$.

1. Matrix of f in $\mathcal{B}$.
Apply f to each basis vector using the binomial expansion $(X+1)^n - X^n$:

$f(1) = 1 - 1 = 0 \implies$ column $(0,0,0,0)^T$.

$f(X) = (X+1) - X = 1 \implies$ column $(1,0,0,0)^T$.

$f(X^2) = (X+1)^2 - X^2 = 2X + 1 \implies$ column $(1,2,0,0)^T$.

$f(X^3) = (X+1)^3 - X^3 = 3X^2 + 3X + 1 \implies$ column $(1,3,3,0)^T$.

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

2. Rank and solvability.
The matrix is upper triangular. Three diagonal entries are non-zero $(1,2,3)$, so $\operatorname{rank}(M) = 3$.
Now, $\mathbb{R}_2[X]$ has dimension 3. The image of f lives inside $\mathbb{R}_3[X]$ (since the degree drops by one), and has dimension 3, which equals $\dim(\mathbb{R}_2[X])$. Therefore $\operatorname{Im}(f) \supseteq \mathbb{R}_2[X]$, and the equation $f(P) = Q$ has at least one solution for every $Q \in \mathbb{R}_2[X]$.

▶ Note

$f(P) = P(X+1) - P(X)$ always *lowers the degree by 1.* That is why the last row of M is all zeros and why the relevant codomain is $\mathbb{R}_2[X]$, not all of $\mathbb{R}_3[X]$.

Exercise 7

We solve for $X \in \mathcal{M}_n(\mathbb{R})$ where $a \in \mathbb{R}$ and A, B are invertible.

(a) $aX + \operatorname{tr}(X)A = B$**.**
Idea: Take the trace of both sides to find $\operatorname{tr}(X)$, then isolate X.
Taking the trace (using $\operatorname{tr}(\lambda M) = \lambda \operatorname{tr}(M)$ and $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$):

$$a \operatorname{tr}(X) + \operatorname{tr}(X) \operatorname{tr}(A) = \operatorname{tr}(B) \implies \operatorname{tr}(X)\big(a + \operatorname{tr}(A)\big) = \operatorname{tr}(B).$$

Assuming $a + \operatorname{tr}(A) \neq 0$:

$$\operatorname{tr}(X) = \frac{\operatorname{tr}(B)}{a + \operatorname{tr}(A)}.$$

Substituting back into the original equation: $aX = B - \operatorname{tr}(X)A$, so:

$$\boxed{X = \frac{1}{a}\left(B - \frac{\operatorname{tr}(B)}{a + \operatorname{tr}(A)}\, A\right).}$$

(b) $aX + \operatorname{tr}(XA)\, I_n = B$**.**
Idea: Multiply the equation by A on the right and take the trace, to isolate $\operatorname{tr}(XA)$.
Multiplying $aX + \operatorname{tr}(XA)I_n = B$ by A on the right:

$$aXA + \operatorname{tr}(XA)\, A = BA.$$

Taking the trace:

$$a \operatorname{tr}(XA) + \operatorname{tr}(XA) \operatorname{tr}(A) = \operatorname{tr}(BA) \implies \operatorname{tr}(XA)\big(a + \operatorname{tr}(A)\big) = \operatorname{tr}(BA).$$

Assuming $a + \operatorname{tr}(A) \neq 0$:

$$\operatorname{tr}(XA) = \frac{\operatorname{tr}(BA)}{a + \operatorname{tr}(A)}.$$

Substituting back: $aX = B - \operatorname{tr}(XA)\, I_n$, so:

$$\boxed{X = \frac{1}{a}\left(B - \frac{\operatorname{tr}(BA)}{a + \operatorname{tr}(A)}\, I_n\right).}$$

(c) $\det(A)\, XB + \operatorname{tr}(XA)\, B = A$**.**
Idea: Factor B out and reduce to case (b).
Factor the left side:

$$\big(\det(A)\, X + \operatorname{tr}(XA)\, I_n\big)B = A.$$

Since B is invertible, multiply on the right by B^{-1}:

$$\det(A)\, X + \operatorname{tr}(XA)\, I_n = AB^{-1}.$$

This is exactly the form of part (b) with $a = \det(A)$ and $B \leftarrow AB^{-1}$. Applying the result of (b) (with $\operatorname{tr}(XA)(\det(A) + \operatorname{tr}(A)) = \operatorname{tr}(AB^{-1}A)$):

$$\boxed{X = \frac{1}{\det(A)}\left(AB^{-1} - \frac{\operatorname{tr}(AB^{-1}A)}{\det(A) + \operatorname{tr}(A)}\, I_n\right).}$$

▶ **Note**

All three cases require $a + \operatorname{tr}(A) \neq 0$ (and $a \neq 0$, $\det(A) \neq 0$ for part (c)). Always state these conditions in your answer. Also recall that $\operatorname{tr}(XA) = \operatorname{tr}(AX)$ by the cyclic property of the trace.

Exercise 8

We study the system:

$$\begin{cases} x - y + 2z = a \\ mx + (1-m)y + (2m-2)z = m \\ 2x + my - (3m+1)z = 2a \end{cases}$$

Step 1 — Form the augmented matrix and reduce.

$$\begin{pmatrix} 1 & -1 & 2 & a \\ m & 1-m & 2m-2 & m \\ 2 & m & -(3m+1) & 2a \end{pmatrix}$$

$L_2 \leftarrow L_2 - mL_1$: the second row becomes

$$\big(0,\ 1-m+m,\ 2m-2-2m,\ m-ma\big) = \big(0,\ 1,\ -2,\ m(1-a)\big).$$

$L_3 \leftarrow L_3 - 2L_1$: the third row becomes

$$\big(0,\ m+2,\ -(3m+1)-4,\ 2a-2a\big) = \big(0,\ m+2,\ -(3m+5),\ 0\big).$$

$L_3 \leftarrow L_3 - (m+2)L_2$:

$$\big(0,\ 0,\ -(3m+5)+2(m+2),\ -(m+2)m(1-a)\big) = \big(0,\ 0,\ -m-1,\ -m(m+2)(1-a)\big).$$

Step 2 — Read the final row: $z(-m-1) = -m(m+2)(1-a)$.

- **Unique solution:** $-m-1 \neq 0$, i.e. $m \neq -1$. Then z is uniquely determined, and back-substitution gives unique y and x.
- **Infinitely many solutions:** $m = -1$ and the RHS also equals zero: $-(-1)(1)(1-a) = (1-a) = 0$, so $a = 1$.
- **No solution:** $m = -1$ and $a \neq 1$ (the last equation becomes $0 = 1 - a \neq 0$).

▶ **Note**

After row reduction, always check *both* the pivot *and* the right-hand side of the critical row. A zero pivot alone does not tell you whether the system is consistent; you must check whether the RHS is also zero.

Exercise 9

The system is:

$$\begin{cases} x + my = -3 \\ mx + 4y = 6 \end{cases}$$

Step 1 — Compute the determinant.

$$D = \begin{vmatrix} 1 & m \\ m & 4 \end{vmatrix} = 4 - m^2 = (2 - m)(2 + m).$$

Step 2 — Three cases.

Case 1: $m \neq \pm 2$ (unique solution). By Cramer's rule:

$$x = \frac{\begin{vmatrix} -3 & m \\ 6 & 4 \end{vmatrix}}{D} = \frac{-12 - 6m}{4 - m^2} = \frac{-6(m+2)}{(2-m)(2+m)} = \frac{-6}{2-m} = \frac{6}{m-2},$$

$$y = \frac{\begin{vmatrix} 1 & -3 \\ m & 6 \end{vmatrix}}{D} = \frac{6 + 3m}{4 - m^2} = \frac{3(m+2)}{(2-m)(2+m)} = \frac{3}{2-m} = \frac{-3}{m-2}.$$

Case 2: $m = 2$ (no solution).
The system becomes $x + 2y = -3$ and $2x + 4y = 6$, i.e. $x + 2y = 3$. These two lines are **parallel** (same direction vector, different constants): they never intersect.

Case 3: $m = -2$ (infinitely many solutions).
The system becomes $x - 2y = -3$ and $-2x + 4y = 6$, i.e. $x - 2y = -3$. Both equations are **identical**: the two lines coincide. The free parameter is $y = t \in \mathbb{R}$, giving $x = 2t - 3$.

Geometric interpretation. Each equation represents a line in the plane $\mathbb{R}^2$.

- The solution is the **intersection** of the two lines.
- $m \neq \pm 2$: the lines are **secant** (one intersection point).
- $m = 2$: the lines are **parallel and distinct** (no intersection).
- $m = -2$: the lines are **coincident** (infinitely many intersections).

▶ Note

In Cramer's rule, cancel common factors *carefully*. Here $(m + 2)$ cancels, but only when $m \neq -2$. Plugging $m = -2$ directly into the simplified formula gives a false result; always handle boundary cases separately.

Exercise 10

We balance each equation by writing unknown integer coefficients and solving the linear system of atom-conservation equations.

1. $x_1\,NH_3 + x_2\,O_2 \longrightarrow x_3\,NO + x_4\,H_2O$
Setting up one equation per element:

- **Nitrogen (N):** $x_1 = x_3$.
- **Hydrogen (H):** $3x_1 = 2x_4 \implies x_4 = \frac{3}{2}x_1$.
- **Oxygen (O):** $2x_2 = x_3 + x_4 = x_1 + \frac{3}{2}x_1 = \frac{5}{2}x_1 \implies x_2 = \frac{5}{4}x_1$.

Choose the free parameter $x_1 = 4$ (LCM of denominators 4 and 2) to get integers:

$$\boxed{4\,NH_3 + 5\,O_2 \rightarrow 4\,NO + 6\,H_2O.}$$

2. $x_1\,C_2H_6 + x_2\,O_2 \longrightarrow x_3\,CO_2 + x_4\,H_2O$

- **Carbon (C):** $2x_1 = x_3 \implies x_3 = 2x_1$.
- **Hydrogen (H):** $6x_1 = 2x_4 \implies x_4 = 3x_1$.
- **Oxygen (O):** $2x_2 = 2x_3 + x_4 = 4x_1 + 3x_1 = 7x_1 \implies x_2 = \frac{7}{2}x_1$.

Choose $x_1 = 2$ to clear the denominator:

$$\boxed{2\,C_2H_6 + 7\,O_2 \rightarrow 4\,CO_2 + 6\,H_2O.}$$

▶ Note

The free parameter must be chosen so that *all* coefficients are **positive integers** with no common factor. It equals the LCM of the denominators that appear in the solution. Fractional coefficients are not physically meaningful.