Fluid Mechanics
Correction – Set 2

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Statics of Fluids (Hydrostatics) — Detailed Correction

Fluid Mechanics — Detailed Correction – Exercise Set 2

Focus on the Way of Thinking, Explicit Substitutions, and Step-by-Step Course Execution

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How to Use This Correction

Each solution follows a strict four-stage analytical structure designed to train exam intuition: **Method** (identifies the exact course technique) ⟶ **Why we do this** (explains the mental train of thought and physical clues) ⟶ **Execution** (complete line-by-line derivation with zero skipped steps and explicit **Sub** callouts) ⟶ **Pitfall** / **Sanity Check** (where students lose points and how to verify your answer).

Reference Constants: $\rho_w = 1000\,\mathrm{kg\,m^{-3}}$, $g = 9.81\,\mathrm{m\,s^{-2}}$, $\gamma_w = \rho_w g = 9810\,\mathrm{N\,m^{-3}}$, $p_{\mathrm{atm}} = 101.325\,\mathrm{kPa}$, $SG_{\mathrm{Hg}} = 13.6$ ($\gamma_{\mathrm{Hg}} = 133\,416\,\mathrm{N\,m^{-3}}$), $R = 8.314\,\mathrm{J\,mol^{-1}\,K^{-1}}$.

Difficulty: ★☆☆☆☆ Direct ★★☆☆☆ Standard ★★★☆☆ Exam level ★★★★☆ Tricky ★★★★★ Synthesis

Exercise 1

Configuration Summary: *Closed tank (floor $z = 0$) connected at the bottom to an open mercury standpipe ($z_2 = 0.65\,\mathrm{m}$, p_{atm}). Tank layers: mercury ($SG_{\mathrm{Hg}} = 13.6$) up to $z_1 = 0.40\,\mathrm{m}$, water of thickness $h_w = 2.50\,\mathrm{m}$ ($z \in [0.40, 2.90]\,\mathrm{m}$), oil ($SG_o = 0.80$) of thickness $h_o = 1.50\,\mathrm{m}$ ($z \in [2.90, 4.40]\,\mathrm{m}$), and trapped air above $z = 4.40\,\mathrm{m}$.*

1. Determine the gauge pressure and the absolute pressure of the trapped air, and express the absolute air pressure as an equivalent mercury head (in mmHg). ★★☆☆☆

Method *Piecewise Hydrostatic Manometer Traversal and Isobaric Plane Jump*

> ***Why we do this (Way of Thinking):***
> The only known boundary pressure is atmospheric pressure at the open standpipe meniscus ($z_2 = 0.65\,\mathrm{m}$, where $p_{\mathrm{gage}} = 0$). Because a continuous body of static mercury connects the standpipe down to the floor and up to the mercury–water interface at $z_1 = 0.40\,\mathrm{m}$ inside the tank, the horizontal plane $z = z_1$ inside the mercury is an isobar. Once $p(z_1)$ is obtained from the standpipe column ($z_2 - z_1$), climbing upward through the water layer h_w and the oil layer h_o subtracts $\gamma_w h_w$ and $\gamma_o h_o$ to isolate the uniform trapped-air pressure.

Execution:

Step 1 — Specific weights and gauge pressure at the $z_1 = 0.40\,\mathrm{m}$ mercury interface:

$$\gamma_w = \rho_w g = (1000\,\mathrm{kg\,m^{-3}})(9.81\,\mathrm{m\,s^{-2}}) = 9810\,\mathrm{N\,m^{-3}},$$
$$\gamma_o = SG_o\,\gamma_w = 0.80 \times 9810\,\mathrm{N\,m^{-3}} = 7848\,\mathrm{N\,m^{-3}},$$
$$\gamma_{\mathrm{Hg}} = SG_{\mathrm{Hg}}\,\gamma_w = 13.6 \times 9810\,\mathrm{N\,m^{-3}} = 133\,416\,\mathrm{N\,m^{-3}}.$$

Along the continuous mercury path from the open meniscus ($z_2 = 0.65\,\mathrm{m}$, $p_{2,\mathrm{gage}} = 0$) down to elevation $z_1 = 0.40\,\mathrm{m}$:

$$p_{1,\mathrm{gage}} = 0 + \gamma_{\mathrm{Hg}}(z_2 - z_1) = (133\,416\,\mathrm{N\,m^{-3}})(0.65 - 0.40)\,\mathrm{m} = 133\,416 \times 0.25 = 33\,354\,\mathrm{Pa}.$$

Step 2 — Gauge and absolute pressure of the trapped air: Integrating the fundamental equation of fluid statics upward from $z_1 = 0.40\,\mathrm{m}$ across the water layer ($h_w = 2.50\,\mathrm{m}$) and the oil layer ($h_o = 1.50\,\mathrm{m}$), while neglecting the hydrostatic weight of the trapped air pocket:

$$\begin{aligned} p_{\mathrm{air,gage}} &= p_{1,\mathrm{gage}} - \gamma_w h_w - \gamma_o h_o \\ &= 33\,354\,\mathrm{Pa} - (9810\,\mathrm{N\,m^{-3}})(2.50\,\mathrm{m}) - (7848\,\mathrm{N\,m^{-3}})(1.50\,\mathrm{m}) \\ &= 33\,354 - 24\,525 - 11\,772 = -2943\,\mathrm{Pa} = -2.943\,\mathrm{kPa}. \end{aligned}$$

Adding standard atmospheric pressure $p_{\text{atm}} = 101\,325\,\text{Pa}$ gives the absolute air pressure:

$$p_{\text{air,abs}} = p_{\text{atm}} + p_{\text{air,gage}} = 101\,325 - 2943 = 98\,382\,\text{Pa} = 98.382\,\text{kPa}.$$

Step 3 — Equivalent mercury head h_{Hg}:

$$h_{\text{Hg}} = \frac{p_{\text{air,abs}}}{\gamma_{\text{Hg}}} = \frac{98\,382\,\text{Pa}}{133\,416\,\text{N m}^{-3}} = 0.737\,408\,\text{m} = 737.4\,\text{mmHg}.$$

$$p_{\text{air,gage}} = -2.943\,\text{kPa}, \qquad p_{\text{air,abs}} = 98.382\,\text{kPa}, \qquad h_{\text{Hg}} = 737.4\,\text{mmHg}$$

► **Common Pitfall:**
A negative gauge pressure ($p_{\text{air,gage}} = -2.943\,\text{kPa}$) indicates a partial vacuum relative to ambient air, not a negative absolute pressure. Dropping the minus sign when evaluating $p_{\text{air,abs}} = p_{\text{atm}} + p_{\text{air,gage}}$ erroneously shifts the absolute pressure above 101.325 kPa.

✓ **Sanity Check:**
Because $p_{\text{air,abs}} < p_{\text{atm}}$, its mercury head must be slightly below the standard sea-level barometric height ($\approx 760\,\text{mmHg}$). Indeed, $737.4\,\text{mmHg} < 760\,\text{mmHg}$, with a deficit of $2943\,\text{Pa}/133\,416\,\text{N m}^{-3} = 22.1\,\text{mmHg}$.

2. Compute the absolute pressure acting on the tank floor ($z = 0$). ★★★★★

Method *Direct Integration Through the Continuous Mercury Column*

Why we do this (Way of Thinking):
We have two equivalent pathways to the floor $z = 0$: descending through all four fluid layers inside the closed tank, or descending directly through the single homogeneous mercury column from the open standpipe surface at $z_2 = 0.65\,\text{m}$ down to $z = 0$. Because mercury fills the bottom of both vessels continuously, the standpipe path reaches $z = 0$ in a single step and eliminates accumulated rounding error.

Execution: Starting at the open standpipe free surface ($z_2 = 0.65\,\text{m}$, $p_2 = p_{\text{atm}}$) and descending to $z = 0$ inside mercury:

$$p_{\text{floor,gage}} = \gamma_{\text{Hg}}\, z_2 = (133\,416\,\text{N m}^{-3})(0.65\,\text{m}) = 86\,720.4\,\text{Pa} = 86.720\,\text{kPa},$$
$$p_{\text{floor,abs}} = p_{\text{atm}} + p_{\text{floor,gage}} = 101\,325 + 86\,720.4 = 188\,045.4\,\text{Pa} \approx 188.05\,\text{kPa}.$$

$$p_{\text{floor,abs}} = 188.045\,\text{kPa} \quad (p_{\text{floor,gage}} = 86.720\,\text{kPa})$$

✓ **Sanity Check:**
Cross-checking downward from the trapped air inside the tank: $p_{\text{floor,abs}} = p_{\text{air,abs}} + \gamma_o h_o + \gamma_w h_w + \gamma_{\text{Hg}} z_1 = 98\,382 + 11\,772 + 24\,525 + (133\,416)(0.40) = 134\,679 + 53\,366.4 = 188\,045.4\,\text{Pa}$, matching identically.

3. Points M (inside the water layer) and N (inside the standpipe mercury column) lie at the same elevation

$z_M = z_N = 0.55\,\text{m}$. **State whether $p_M = p_N$, justify your answer, and compute $p_M - p_N$.** ★★☆☆☆

Method *Isobaric Connectivity Criterion and Differential Column Comparison*

> ***Why we do this (Way of Thinking):***
> Horizontal planes are isobaric ($p_M = p_N$ at $z_M = z_N$) if and only if M and N can be connected by a continuous path lying entirely within a single homogeneous static fluid. Here, any path through the bottom channel from M (at $z = 0.55\,\text{m}$ in water) to N (at $z = 0.55\,\text{m}$ in mercury) must cross the water–mercury interface at $z_1 = 0.40\,\text{m}$. Starting from the highest common isobar $z_1 = 0.40\,\text{m}$, climbing $\Delta z = 0.15\,\text{m}$ in water reduces pressure much less than climbing $\Delta z = 0.15\,\text{m}$ in dense mercury, so $p_M > p_N$.

Execution: Because no continuous single-fluid path joins M and N, $p_M \neq p_N$. At the common mercury isobar $z_1 = 0.40\,\text{m}$, both branches share the exact pressure p_1 ($p_{1,\text{gage}} = 33\,354\,\text{Pa}$). Climbing from $z_1 = 0.40\,\text{m}$ to $z_M = z_N = 0.55\,\text{m}$ ($\Delta z = z_M - z_1 = 0.15\,\text{m}$) in each branch separately:

$$p_{M,\text{gage}} = p_{1,\text{gage}} - \gamma_w(z_M - z_1) = 33\,354 - (9810)(0.15) = 33\,354 - 1471.5 = 31\,882.5\,\text{Pa},$$
$$p_{N,\text{gage}} = p_{1,\text{gage}} - \gamma_{\text{Hg}}(z_N - z_1) = 33\,354 - (133\,416)(0.15) = 33\,354 - 20\,012.4 = 13\,341.6\,\text{Pa}.$$

Subtracting p_N from p_M (noting that p_{atm} cancels between absolute and gauge differences):

$$p_M - p_N = p_{M,\text{gage}} - p_{N,\text{gage}} = (\gamma_{\text{Hg}} - \gamma_w)(z_M - z_1) = (133\,416 - 9810)(0.15) = 123\,606 \times 0.15 = 18\,540.9\,\text{Pa}.$$

$$\boxed{p_M \neq p_N \quad \text{(different fluids across } z > z_1\text{)}, \qquad p_M - p_N = +18.541\,\text{kPa}}$$

> ▸ **Common Pitfall:**
> Blindly writing "same height implies same pressure" across separate limbs of a multi-fluid system is the most frequent hydrostatics exam error. Always locate the lowest/highest interface bounding a continuous single fluid ($z_1 = 0.40\,\text{m}$ here) before equating pressures horizontally.

Exercise 2

Configuration Summary: *Ideal-gas atmosphere with molar mass $M = 29.0\,\text{g mol}^{-1} = 0.0290\,\text{kg mol}^{-1}$, sea-level conditions $z_0 = 0$, $p_0 = 101.325\,\text{kPa}$, $T_0 = 288\,\text{K}$ (isothermal), uniform $g = 9.81\,\text{m s}^{-2}$, and $R = 8.314\,\text{J mol}^{-1}\,\text{K}^{-1}$.*

1. **Starting from the fundamental equation of fluid statics and the ideal-gas equation of state, derive the pressure distribution $p(z)$ as a function of altitude z for an isothermal atmosphere at temperature T_0.** ★★☆☆☆

Method *Separation of Variables in the Compressible Hydrostatic ODE*

> ***Why we do this (Way of Thinking):***
> Unlike a liquid, atmospheric air is compressible: its density ρ decreases as pressure p drops with altitude, so we cannot integrate $dp/dz = -\rho g$ by treating ρ as a constant. Expressing ρ in terms of p via the ideal-gas equation of state turns $dp/dz = -\rho g$ into a separable first-order differential equation in (p, z).

Execution: For n moles of ideal gas of mass $m = nM$ occupying volume V at absolute temperature T_0, the equation of state reads

$$pV = nRT_0 = \frac{m}{M}RT_0 \implies \rho = \frac{m}{V} = \frac{pM}{RT_0}.$$

Substituting this density law into the vertical component of the fundamental equation of fluid statics (z-axis

oriented upward), $\frac{dp}{dz} = -\rho g$, yields:

$$\frac{dp}{dz} = -\frac{Mg}{RT_0}\,p.$$

Sub **Separate variables for $p > 0$: $\frac{dp}{p} = -\frac{Mg}{RT_0}\,dz$. Integrate from sea level ($z_0 = 0$, $p = p_0$) to altitude (z, $p(z)$):**

$$\int_{p_0}^{p(z)} \frac{dp'}{p'} = -\frac{Mg}{RT_0}\int_0^z dz'$$
$$\ln\big(p(z)\big) - \ln(p_0) = -\frac{Mg}{RT_0}\,(z-0)$$
$$\ln\left(\frac{p(z)}{p_0}\right) = -\frac{Mgz}{RT_0}.$$

Exponentiating both sides produces the barometric law:

$$\boxed{p(z) = p_0 \exp\left(-\frac{Mgz}{RT_0}\right)}$$

> ► **Common Pitfall:**
> Molar mass is given as $M = 29.0\,\mathrm{g\,mol^{-1}}$; leaving M in $\mathrm{g\,mol^{-1}}$ instead of converting to base SI units ($M = 0.0290\,\mathrm{kg\,mol^{-1}}$) inflates the dimensionless exponent $Mgz/(RT_0)$ by a factor of 1000.

2. Compute the atmospheric pressure and air density at altitude $z_1 = 4000\,\mathrm{m}$. ★★☆☆☆

Method ***Barometric Exponential Evaluation and Ideal-Gas Density Law***

> ***Why we do this (Way of Thinking):***
> Evaluating the dimensionless ratio $Mgz_1/(RT_0)$ first (or equivalently the atmospheric scale height $H_s = RT_0/(Mg)$) provides an immediate dimensional check before computing $p(z_1)$ and deducing $\rho(z_1) = p(z_1)M/(RT_0)$.

Execution: Converting $M = 0.0290\,\mathrm{kg\,mol^{-1}}$, the dimensionless exponent at $z_1 = 4000\,\mathrm{m}$ is:

$$\frac{Mgz_1}{RT_0} = \frac{(0.0290\,\mathrm{kg\,mol^{-1}})(9.81\,\mathrm{m\,s^{-2}})(4000\,\mathrm{m})}{(8.314\,\mathrm{J\,mol^{-1}\,K^{-1}})(288\,\mathrm{K})} = \frac{1137.96}{2394.432} = 0.475253.$$

Evaluating $p(z_1)$ and $\rho(z_1)$:

$$p(z_1) = (101.325\,\mathrm{kPa})\,e^{-0.475253} = 101.325 \times 0.621728 = 62.997\,\mathrm{kPa} \quad (62\,996.6\,\mathrm{Pa}),$$
$$\rho(z_1) = \frac{p(z_1)\,M}{RT_0} = \frac{(62\,996.6\,\mathrm{Pa})(0.0290\,\mathrm{kg\,mol^{-1}})}{(8.314\,\mathrm{J\,mol^{-1}\,K^{-1}})(288\,\mathrm{K})} = \frac{1826.901}{2394.432} = 0.7630\,\mathrm{kg\,m^{-3}}.$$

$$\boxed{p(4000\,\mathrm{m}) = 63.00\,\mathrm{kPa}, \qquad \rho(4000\,\mathrm{m}) = 0.763\,\mathrm{kg\,m^{-3}}}$$

> ✓ **Sanity Check:**
> At constant temperature T_0, $\rho(z)/\rho_0 = p(z)/p_0 = e^{-0.475253} \approx 0.6217$, confirming that both pressure and density retain about 62.2% of their sea-level values at 4000 m.

3. Recompute the pressure at $z_1 = 4000\,\text{m}$ under the assumption of an incompressible atmosphere having uniform sea-level density ρ_0, and determine the resulting relative error (in %). ★★☆☆☆

Method *Linear Incompressible Approximation vs. Exact Compressible Law*

> ***Why we do this (Way of Thinking):***
> Holding density artificially fixed at its maximum sea-level value ρ_0 overestimates the weight of every air layer above $z = 0$, causing the linear formula $p_{\text{inc}}(z) = p_0 - \rho_0 g z$ to underpredict the true pressure at high altitude. Note that $p_{\text{inc}}(z_1)$ is simply the two-term first-order Maclaurin truncation $p_0(1-u)$ of $p_0 e^{-u}$ with $u = Mgz_1/(RT_0)$.

Execution: First compute the sea-level air density ρ_0:

$$\rho_0 = \frac{p_0 M}{RT_0} = \frac{(101\,325\,\text{Pa})(0.0290\,\text{kg mol}^{-1})}{2394.432\,\text{J mol}^{-1}} = 1.227\,19\,\text{kg m}^{-3}.$$

Under the uniform-density assumption ($\rho = \rho_0 = \text{const}$), integrating $dp/dz = -\rho_0 g$ yields:

$$\begin{aligned} p_{\text{inc}}(z_1) &= p_0 - \rho_0 g z_1 = 101\,325\,\text{Pa} - (1.227\,19\,\text{kg m}^{-3})(9.81\,\text{m s}^{-2})(4000\,\text{m}) \\ &= 101\,325 - 48\,154.97 = 53\,170.0\,\text{Pa} = 53.170\,\text{kPa}. \end{aligned}$$

Comparing against the compressible isothermal value $p(z_1) = 62\,996.6\,\text{Pa}$:

$$\varepsilon_{\text{rel}} = \frac{|p_{\text{inc}}(z_1) - p(z_1)|}{p(z_1)} \times 100\% = \frac{|53\,170.0 - 62\,996.6|}{62\,996.6} \times 100\% = \frac{9826.6}{62\,996.6} \times 100\% = 15.60\%.$$

$$\boxed{p_{\text{inc}}(4000\,\text{m}) = 53.17\,\text{kPa}, \qquad \varepsilon_{\text{rel}} = 15.60\% \quad \text{(underestimate of 9.83 kPa)}}$$

> ✓ **Sanity Check:**
> Using the Maclaurin identity directly with $u = 0.475253$: $(e^{-u} - (1-u))/e^{-u} = 1 - (1 - 0.475253)/0.621728 = 1 - 0.84401 = 0.1560 = 15.60\%$, independent of p_0.

4. Determine the mercury column height h (in mm) indicated by a mercury barometer of negligible mercury vapor pressure at altitude $z_1 = 4000\,\text{m}$. ★☆☆☆☆

Method *Mercury Barometer Equilibrium with Evacuated Torricelli Space*

> ***Why we do this (Way of Thinking):***
> In a barometer with negligible mercury vapor pressure, the closed top apex is at absolute vacuum ($p_{\text{top,abs}} = 0$), while the open pool surface experiences the local atmospheric pressure $p(z_1)$. Equating pressures at the pool surface level gives $p(z_1) = \gamma_{\text{Hg}} h$.

Execution: With $p(z_1) = 62\,996.6\,\text{Pa}$ and $\gamma_{\text{Hg}} = SG_{\text{Hg}}\,\gamma_w = 133\,416\,\text{N m}^{-3}$:

$$h = \frac{p(z_1)}{\gamma_{\text{Hg}}} = \frac{62\,996.6\,\text{Pa}}{133\,416\,\text{N m}^{-3}} = 0.472\,18\,\text{m} = 472.2\,\text{mm}.$$

$$\boxed{h = 472.2\,\text{mm}}$$

> ▸ **Common Pitfall:**
> A mercury barometer measures **local absolute atmospheric pressure**, never gauge pressure. Using $p_{\text{inc}}(z_1)$ or sea-level p_0 instead of $p(z_1) = 62.997\,\text{kPa}$ produces an incorrect barometric reading.

Exercise 3

Configuration Summary: *Water suction pipe A ($z_A = 1.20\,\text{m}$). Left branch: mercury U-tube ($SG_{\text{Hg}} = 13.6$) with vertical-leg water–mercury interface at $z_1 = 0.80\,\text{m}$ and open outer leg inclined at $\alpha = 30^\circ$ to the horizontal, meniscus at slant distance $L = 1.20\,\text{m}$ below $z = z_1$. Right branch: inverted oil U-tube ($SG_o = 0.80$, interfaces $z_2 = 2.00\,\text{m}$ and $z_3 = 1.00\,\text{m}$) leading to a closed evacuated spherical bulb (center $z_B = 2.20\,\text{m}$, radius $R = 0.30\,\text{m}$) completely filled with liquid SG_x of negligible vapor pressure.*

1. Determine the gauge pressure and the absolute pressure at the pipe centerline A.

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Method *Inclined-Tube Vertical Projection and Manometer Traversal*

> ***Why we do this (Way of Thinking):***
> Hydrostatic pressure variations depend strictly on **vertical** elevation changes Δz, never on the slant tube length L itself. Projecting the slant distance $L = 1.20\,\text{m}$ via $\Delta z_{\text{Hg}} = L\sin\alpha$ locates the vertical elevation of the open mercury meniscus below z_1. Traversing from that open meniscus ($p_{\text{gage}} = 0$) up through mercury to z_1 and then up through water from z_1 to z_A determines p_A directly.

Execution:

Step 1 — Vertical mercury differential height and meniscus elevation z_m: Because the inclined tube makes angle $\alpha = 30^\circ$ with the horizontal, a slant distance $L = 1.20\,\text{m}$ below the plane $z = z_1 = 0.80\,\text{m}$ corresponds to a vertical drop:

$$\Delta z_{\text{Hg}} = z_1 - z_m = L\sin\alpha = (1.20\,\text{m})\sin 30^\circ = 1.20\times 0.50 = 0.60\,\text{m} \implies z_m = 0.80 - 0.60 = 0.20\,\text{m}.$$

Step 2 — Gauge and absolute pressure at A ($z_A = 1.20\,\text{m}$): Starting at the open mercury meniscus ($z_m = 0.20\,\text{m}$, $p_{m,\text{gage}} = 0$), traversing through the continuous mercury bend to $z_1 = 0.80\,\text{m}$ gives a net upward climb of $\Delta z_{\text{Hg}} = 0.60\,\text{m}$ in mercury, followed by an upward climb of $z_A - z_1 = 1.20 - 0.80 = 0.40\,\text{m}$ in water:

$$\begin{aligned} p_{A,\text{gage}} &= 0 - \gamma_{\text{Hg}}(z_1 - z_m) - \gamma_w(z_A - z_1) \\ &= -(133\,416\,\text{N m}^{-3})(0.60\,\text{m}) - (9810\,\text{N m}^{-3})(0.40\,\text{m}) \\ &= -80\,049.6 - 3924.0 = -83\,973.6\,\text{Pa} = -83.974\,\text{kPa}. \end{aligned}$$

Converting to absolute pressure with $p_{\text{atm}} = 101\,325\,\text{Pa}$:

$$p_{A,\text{abs}} = p_{\text{atm}} + p_{A,\text{gage}} = 101\,325 - 83\,973.6 = 17\,351.4\,\text{Pa} = 17.351\,\text{kPa}.$$

$$\boxed{p_{A,\text{gage}} = -83.974\,\text{kPa}, \qquad p_{A,\text{abs}} = 17.351\,\text{kPa}}$$

> ► **Common Pitfall:**
> Two traps frequently appear in inclined-leg suction manometers: (1) multiplying γ_{Hg} by the slant length $L = 1.20\,\text{m}$ instead of the vertical projection $L\sin 30^\circ = 0.60\,\text{m}$, and (2) adding $+\gamma_{\text{Hg}}\Delta z_{\text{Hg}}$ instead of subtracting when the open meniscus lies *below* the internal interface z_1.

2. Determine the specific gravity SG_x of the liquid filling the spherical bulb.

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Method *Absolute-Pressure Traversal to the Top Apex of an Evacuated Bulb*

> ***Why we do this (Way of Thinking):***
> Because the closed spherical bulb is completely filled with liquid of negligible vapor pressure (no trapped air), the liquid cannot sustain negative absolute tension: at the very highest point of the liquid domain—the north pole apex of the sphere at $z_{\text{top}} = z_B + R$—the absolute pressure reaches the vapor/vacuum limit $p_{\text{abs}}(z_{\text{top}}) = 0$. Therefore, we must work in **absolute pressure** and traverse from A through water and the inverted oil arch up to $z_{\text{top}} = z_B + R$.

Execution:

Step 1 — Absolute pressure at the oil–liquid interface $z_3 = 1.00\,\text{m}$: Starting from $p_{A,\text{abs}} = 17\,351.4\,\text{Pa}$ at $z_A = 1.20\,\text{m}$, we climb upward in water to $z_2 = 2.00\,\text{m}$ ($\Delta z_w = 2.00 - 1.20 = 0.80\,\text{m}$), then traverse the continuous oil arch from $z_2 = 2.00\,\text{m}$ to $z_3 = 1.00\,\text{m}$ (intermediate crest above z_2 cancels identically, leaving a net downward drop $\Delta z_o = z_2 - z_3 = 1.00\,\text{m}$ in oil of $\gamma_o = 0.80\,\gamma_w = 7848\,\text{N m}^{-3}$):

$$\begin{aligned} p_{3,\text{abs}} &= p_{A,\text{abs}} - \gamma_w(z_2 - z_A) + \gamma_o(z_2 - z_3) \\ &= 17\,351.4\,\text{Pa} - (9810\,\text{N m}^{-3})(0.80\,\text{m}) + (7848\,\text{N m}^{-3})(1.00\,\text{m}) \\ &= 17\,351.4 - 7848.0 + 7848.0 = 17\,351.4\,\text{Pa}. \end{aligned}$$

(Notice that the 0.80 m rise in water and the 1.00 m drop in $SG_o = 0.80$ oil cancel exactly because $1.00 \times 0.80 = 0.80 \times 1.00$.)

Step 2 — Imposing $p_{\text{abs}}(z_{\text{top}}) = 0$ at the top apex of the sphere: The highest point occupied by liquid SG_x inside the spherical bulb is its top apex:

$$z_{\text{top}} = z_B + R = 2.20 + 0.30 = 2.50\,\text{m}.$$

Traversing through the continuous liquid SG_x ($\gamma_x = SG_x\,\gamma_w$) from $z_3 = 1.00\,\text{m}$ down around the lower bend and up to $z_{\text{top}} = 2.50\,\text{m}$ gives a net upward climb $z_{\text{top}} - z_3 = 2.50 - 1.00 = 1.50\,\text{m}$:

$$p_{3,\text{abs}} - (SG_x\,\gamma_w)(z_{\text{top}} - z_3) = p_{\text{abs}}(z_{\text{top}}) = 0$$

$$SG_x = \frac{p_{3,\text{abs}}}{\gamma_w(z_B + R - z_3)} = \frac{17\,351.4\,\text{Pa}}{(9810\,\text{N m}^{-3})(1.50\,\text{m})} = \frac{17\,351.4}{14\,715.0} = 1.17916.$$

$$\boxed{SG_x = 1.179 \quad (\rho_x \approx 1179\,\text{kg m}^{-3})}$$

> **✓ Sanity Check:**
> At the sphere center $z_B = 2.20\,\text{m}$, the absolute pressure is $p_B = \gamma_x R = (1.17916)(9810)(0.30) = 3470.3\,\text{Pa} > 0$, confirming that absolute pressure remains strictly non-negative throughout the entire spherical volume $z \in [z_B - R,\ z_B + R]$.

3. Compute the error committed on SG_x if the zero-pressure condition inside the bulb were imposed at the sphere center z_B rather than at the highest point of the liquid. ★★☆☆☆

Method *Sensitivity and Error Analysis of the Vacuum Boundary Elevation*

> ***Why we do this (Way of Thinking):***
> Setting $p_{\text{abs}}(z_B) = 0$ at $z_B = 2.20\,\text{m}$ would imply negative absolute pressure $p_{\text{abs}} = -\gamma_x R < 0$ in the upper hemisphere $z \in (z_B,\ z_B + R]$ and shortens the liquid column denominator from $z_B + R - z_3 = 1.50\,\text{m}$ to $z_B - z_3 = 1.20\,\text{m}$.

Execution: Imposing $p_{\text{abs}} = 0$ erroneously at $z_B = 2.20\,\text{m}$ gives the false specific gravity $\widetilde{SG}_x$:

$$\widetilde{SG}_x = \frac{p_{3,\text{abs}}}{\gamma_w(z_B - z_3)} = \frac{17\,351.4\,\text{Pa}}{(9810\,\text{N m}^{-3})(2.20 - 1.00)\,\text{m}} = \frac{17\,351.4}{11\,772.0} = 1.47396.$$

Computing the absolute error $\Delta SG_x = \widetilde{SG}_x - SG_x$ and relative percentage error:

$$\Delta SG_x = 1.47396 - 1.17916 = +0.2948,$$

$$\varepsilon_{\text{rel}} = \frac{\left|\widetilde{SG}_x - SG_x\right|}{SG_x} \times 100\% = \left(\frac{z_B + R - z_3}{z_B - z_3} - 1\right) \times 100\% = \left(\frac{1.50}{1.20} - 1\right) \times 100\% = 25.00\%.$$

$$\boxed{\widetilde{SG}_x = 1.474, \qquad \Delta SG_x = +0.295 \quad (\text{exact } +25.0\% \text{ overestimate})}$$

> **► Common Pitfall:**
> Whenever a closed bulb or dome of finite radius R is completely filled with liquid under vacuum/vapor equilibrium, omitting R in $z_{\text{top}} = z_B + R$ introduces a substantial systematic error (25% here).

Exercise 4

Configuration Summary: *Oil-filled hydraulic lift ($SG_o = 0.88$, $\gamma_o = 0.88 \times 9810 = 8632.8\,\mathrm{N\,m^{-3}}$) with vertical cylinders of diameters $d_1 = 5.0\,\mathrm{cm} = 0.050\,\mathrm{m}$ and $d_2 = 25.0\,\mathrm{cm} = 0.250\,\mathrm{m}$, massless frictionless pistons exposed externally to p_{atm}, and vehicle mass $M = 1500\,\mathrm{kg}$ on piston 2.*

1. When the lower faces of both pistons lie at the same horizontal elevation, determine the vertical force F_0 that must be applied to piston 1 to hold the vehicle in equilibrium. ★☆☆☆☆

Method ***Pascal's Principle at Equal Piston Elevation***

> ***Why we do this (Way of Thinking):***
> When both piston faces sit at the same elevation inside the continuous static oil volume, the horizontal plane joining them is an isobar: $p_{1,\mathrm{gage}} = p_{2,\mathrm{gage}}$. Atmospheric pressure acts on both outer faces and cancels out, yielding $F_0/A_1 = Mg/A_2$, where the area ratio $(A_1/A_2) = (d_1/d_2)^2$ scales quadratically with diameter.

Execution: With $W = Mg = (1500\,\mathrm{kg})(9.81\,\mathrm{m\,s^{-2}}) = 14\,715\,\mathrm{N}$ and piston cross-sectional areas $A_1 = \pi d_1^2/4$, $A_2 = \pi d_2^2/4$:

$$\frac{F_0}{A_1} = \frac{Mg}{A_2} \implies F_0 = \frac{A_1}{A_2}\,Mg = \left(\frac{d_1}{d_2}\right)^2 Mg = \left(\frac{5.0\,\mathrm{cm}}{25.0\,\mathrm{cm}}\right)^2 (14\,715\,\mathrm{N}) = \left(\frac{1}{5}\right)^2 (14\,715) = \frac{14\,715}{25} = 588.6\,\mathrm{N}.$$

$$\boxed{F_0 = 588.6\,\mathrm{N}}$$

> ✓ **Sanity Check:**
> A 5:1 diameter ratio provides a 25:1 force multiplication advantage, reducing a 14.715 kN vehicle weight to under 600 N.

2. Piston 1 is pushed downward by $x_1 = 25.0\,\mathrm{cm}$ from the initial level. Express the upward displacement x_2 of piston 2 in terms of x_1, d_1, and d_2, and compute its numerical value. ★☆☆☆☆

Method ***Incompressible Volume Conservation*** *($A_1x_1 = A_2x_2$)*

> ***Why we do this (Way of Thinking):***
> Because the hydraulic oil is incompressible and completely enclosed between the two tight pistons, the volume $V_{\mathrm{disp}} = A_1x_1$ swept downward in cylinder 1 must equal the volume A_2x_2 swept upward in cylinder 2.

Execution: Equating displaced oil volumes between cylinders 1 and 2:

$$A_1x_1 = A_2x_2 \implies \frac{\pi d_1^2}{4}\,x_1 = \frac{\pi d_2^2}{4}\,x_2 \implies x_2 = \left(\frac{d_1}{d_2}\right)^2 x_1.$$

Substituting $d_1 = 5.0\,\mathrm{cm}$, $d_2 = 25.0\,\mathrm{cm}$, and $x_1 = 25.0\,\mathrm{cm}$:

$$x_2 = \left(\frac{5.0}{25.0}\right)^2 (25.0\,\mathrm{cm}) = \frac{1}{25} \times 25.0\,\mathrm{cm} = 1.00\,\mathrm{cm} = 0.0100\,\mathrm{m}.$$

$$\boxed{x_2 = \left(\frac{d_1}{d_2}\right)^2 x_1 = 1.00\,\mathrm{cm} \quad (0.0100\,\mathrm{m})}$$

> ✓ **Sanity Check:**
> Ideal equal-level mechanical work is conserved: $F_0x_1 = (588.6\,\mathrm{N})(0.250\,\mathrm{m}) = 147.15\,\mathrm{J}$ equals $Wx_2 = (14\,715\,\mathrm{N})(0.0100\,\mathrm{m}) = 147.15\,\mathrm{J}$.

3. Determine the vertical force F_1 required on piston 1 to maintain equilibrium in this displaced configuration. ★★★☆☆

Method ***Hydrostatic Elevation Correction Between Unequal Piston Levels***

> ***Why we do this (Way of Thinking):***
> Once piston 1 moves down by x_1 below the initial level and piston 2 rises by x_2 above the initial level, the two piston faces no longer lie on the same isobar: piston 1 sits at a total vertical depth $\Delta h = x_1 + x_2$ *below* piston 2. By the fundamental equation of hydrostatics, the gauge pressure under piston 1 must exceed the gauge pressure under piston 2 by $\gamma_o(x_1 + x_2)$, requiring an extra force $\Delta F = \gamma_o(x_1 + x_2)A_1$ on piston 1.

Execution:

Step 1 — Total vertical level difference and piston-1 area:

$$\Delta h = x_1 + x_2 = 0.250 + 0.010 = 0.260\,\text{m}, \qquad A_1 = \frac{\pi d_1^2}{4} = \frac{\pi(0.050\,\text{m})^2}{4} = 1.963\,50 \times 10^{-3}\,\text{m}^2.$$

Step 2 — Hydrostatic balance between the lower faces of pistons 1 and 2: With $p_{1,\text{gage}} = F_1/A_1$, $p_{2,\text{gage}} = Mg/A_2$, and $\gamma_o = SG_o\,\gamma_w = 0.88 \times 9810 = 8632.8\,\text{N m}^{-3}$:

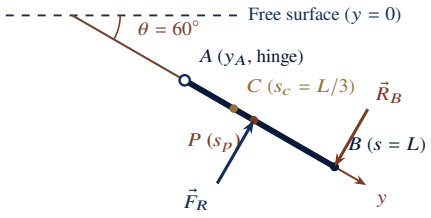
$$\begin{aligned} p_{1,\text{gage}} &= p_{2,\text{gage}} + \gamma_o(x_1 + x_2) \\ \frac{F_1}{A_1} &= \frac{Mg}{A_2} + \gamma_o(x_1 + x_2) \\ F_1 &= \underbrace{\frac{A_1}{A_2}\,Mg}_{F_0} + \gamma_o(x_1 + x_2)A_1 \\ &= 588.600\,\text{N} + (8632.8\,\text{N m}^{-3})(0.260\,\text{m})(1.963\,50 \times 10^{-3}\,\text{m}^2) \\ &= 588.600 + 4.407 = 593.01\,\text{N}. \end{aligned}$$

$$\boxed{F_1 = F_0 + \gamma_o(x_1 + x_2)A_1 = 593.0\,\text{N}}$$

> **► Common Pitfall:**
> Two mistakes commonly occur here: (1) using only $x_1 = 0.25\,\text{m}$ instead of the full differential column height $\Delta h = x_1 + x_2 = 0.26\,\text{m}$, or (2) subtracting $\gamma_o \Delta h$ instead of adding it (since piston 1 is *lower* than piston 2, $p_1 > p_2$, so piston 1 must support both the load pressure and the 0.26 m oil column).

Exercise 5

Configuration Summary: *Planar isosceles triangular gate AB inclined at $\theta = 60°$ to the horizontal, top horizontal base $b = 1.80\,\text{m}$ hinged along A at depth $h_A = 1.50\,\text{m}$, slant altitude $L = 2.40\,\text{m}$ down to bottom vertex B resting on a smooth normal stop (R_B), outer face at p_{atm}, negligible gate mass.*



1. Assuming the airspace above the water is open to the atmosphere, determine the magnitude F_R of the resultant hydrostatic force exerted by the water on the gate, the slant distance from hinge A to the center

of pressure, and the stop reaction R_B. ★★★☆☆

Method ***Submerged Plane Surface Theorems ($F_R = \gamma h_c A$, $y_p = y_c + I_{xc}/(y_c A)$) and Hinge Moment Balance***

> ***Why we do this (Way of Thinking):***
> Let y measure slant distance along the inclined wall plane from the free-surface intersection line O ($y = 0$, $h = 0$), and let $s = y - y_A \in [0, L]$ measure slant distance downward along the gate from the top horizontal hinge base A. Because the triangle's horizontal base of width b sits at the **top** (A) and its apex points **downward** (B), the centroid C lies at one-third of the altitude from the base A: $s_c = L/3$ (not $2L/3$). Once F_R and $s_p = y_p - y_A$ are computed, taking moments about hinge A eliminates the unknown hinge reactions immediately.

Execution:

Step 1 — Gate area, centroid coordinates (y_c, h_c), and centroidal moment of inertia I_{xc}: With $\sin\theta = \sin 60^\circ = \dfrac{\sqrt{3}}{2} \approx 0.866025$:

$$A = \frac{1}{2} bL = \frac{1}{2}(1.80\,\text{m})(2.40\,\text{m}) = 2.160\,\text{m}^2,$$
$$I_{xc} = \frac{bL^3}{36} = \frac{(1.80\,\text{m})(2.40\,\text{m})^3}{36} = \frac{1.80 \times 13.824}{36} = 0.6912\,\text{m}^4,$$
$$y_A = \frac{h_A}{\sin\theta} = \frac{1.50\,\text{m}}{\sin 60^\circ} = \sqrt{3}\,\text{m} \approx 1.732\,05\,\text{m}.$$

Since the horizontal base b is at A, the centroid lies at slant distance $s_c = L/3 = 2.40/3 = 0.800\,\text{m}$ below A:

$$y_c = y_A + s_c = 1.73205 + 0.800 = 2.532\,05\,\text{m},$$
$$h_c = y_c \sin\theta = h_A + s_c \sin 60^\circ = 1.50 + 0.800 \times 0.866025 = 2.192\,82\,\text{m}.$$

Step 2 — Resultant hydrostatic force F_R: Atmospheric pressure on the free surface and on the outer gate face cancels, leaving:

$$F_R = \gamma_w h_c A = (9810\,\text{N m}^{-3})(2.192\,82\,\text{m})(2.160\,\text{m}^2) = 46\,465.0\,\text{N} = 46.465\,\text{kN}.$$

Step 3 — Slant distance s_p from hinge A to the center of pressure P: Applying $y_p = y_c + \dfrac{I_{xc}}{y_c A}$ and subtracting y_A (noting $y_c - y_A = s_c = 0.800\,\text{m}$):

$$s_p = y_p - y_A = s_c + \frac{I_{xc}}{y_c A} = 0.800\,\text{m} + \frac{0.6912\,\text{m}^4}{(2.532\,05\,\text{m})(2.160\,\text{m}^2)}$$
$$= 0.800 + \frac{0.6912}{5.46923} = 0.800 + 0.12638 = 0.9264\,\text{m}.$$

Step 4 — Rotational equilibrium about hinge A ($\sum M_A = 0$): Both $\vec{F}_R$ (at lever arm s_p from A) and $\vec{R}_B$ (at lever arm $L = 2.40\,\text{m}$ from A) act perpendicular to the planar gate in opposite senses:

$$R_B L - F_R s_p = 0 \implies R_B = \frac{s_p}{L} F_R = \frac{0.926\,38\,\text{m}}{2.40\,\text{m}} (46\,465.0\,\text{N}) = 17\,935.1\,\text{N} = 17.935\,\text{kN}.$$

$$F_R = 46.47\,\text{kN}, \qquad s_p = 0.9264\,\text{m from } A, \qquad R_B = 17.94\,\text{kN}$$

> ► **Common Pitfall:**
> Two frequent geometry errors occur on triangular gates: (1) placing the centroid at $2L/3$ from A instead of $L/3$ from the horizontal base A, and (2) using $I_{xc} = bL^3/12$ (rectangle or base-axis triangle formula) instead of the centroidal triangle value $I_{xc} = bL^3/36$.

2. The reservoir roof is now sealed and the trapped airspace is pressurized to a gauge reading $p_0 = 19.62\,\text{kPa}$ while the outer face of the gate remains at atmospheric pressure. Determine the new resultant net force on the gate, the slant distance from hinge A to its point of application, and the new stop reaction R'_B.

★★★★☆

Method *Superposition of Uniform Gas Overpressure (or Equivalent Piezometric Free-Surface Shift)*

> ***Why we do this (Way of Thinking):***
> Sealing and pressurizing the airspace to gauge pressure $p_0 = 19.62\,\text{kPa}$ adds a uniform normal pressure p_0 to every point of the submerged gate while the dry outer face remains at $p_{\text{gage}} = 0$. We can solve this cleanly by **force superposition**: the total net force F'_R is the sum of the open-tank water force F_R (acting at s_p) and a uniform-pressure force $F_0 = p_0 A$ (acting directly at the area centroid $s_c = L/3$). Equivalently, raising the virtual free surface by $h_0 = p_0/\gamma_w = 2.00\,\text{m}$ yields the exact same result.

Execution:

Step 1 — Additional uniform-pressure force F_0 and new resultant force F'_R: With $p_0 = 19\,620\,\text{Pa}$ (corresponding to an exact equivalent water head $h_0 = p_0/\gamma_w = 19\,620/9810 = 2.000\,\text{m}$):

$$F_0 = p_0 A = (19\,620\,\text{Pa})(2.160\,\text{m}^2) = 42\,379.2\,\text{N} = 42.379\,\text{kN},$$
$$F'_R = F_R + F_0 = 46\,465.0 + 42\,379.2 = 88\,844.2\,\text{N} = 88.844\,\text{kN}.$$

Step 2 — New center of pressure s'_p from hinge A (by moment equivalence about A): Since F_R acts at $s_p = 0.926\,38\,\text{m}$ and the uniform force F_0 acts at the geometric centroid $s_c = 0.800\,\text{m}$, equating moments $F'_R s'_p = F_R s_p + F_0 s_c$ gives:

$$s'_p = \frac{F_R s_p + F_0 s_c}{F'_R} = \frac{(46\,465.0)(0.92638) + (42\,379.2)(0.800)}{88\,844.2}$$
$$= \frac{43\,044.22 + 33\,903.36}{88\,844.2} = \frac{76\,947.58}{88\,844.2} = 0.8661\,\text{m}.$$

Step 3 — New stop reaction R'_B at vertex B:

$$R'_B = \frac{F'_R s'_p}{L} = \frac{76\,947.58\,\text{N m}}{2.40\,\text{m}} = 32\,061.5\,\text{N} = 32.061\,\text{kN}.$$

$$F'_R = 88.84\,\text{kN}, \qquad s'_p = 0.8661\,\text{m from } A, \qquad R'_B = 32.06\,\text{kN}$$

> ► **Common Pitfall:**
> Do not apply $y_p = y_c + I_{xc}/(y_c A)$ using the unshifted physical slant depth $y_c = 2.532\,\text{m}$ when the free surface is pressurized ($p_0 \neq 0$): that formula assumes $p_{\text{gage}} = 0$ at $y = 0$. If you use the formula directly, you must measure $y'_c = (h_c + p_0/\gamma_w)/\sin\theta$ from the *virtual* zero-gauge-pressure plane.

> ✓ **Sanity Check:**
> Verification via the virtual free-surface method ($h_0 = 2.000\,\text{m}$): $y'_c = y_c + h_0/\sin 60° = 2.53205 + 2.30940 = 4.841\,45\,\text{m}$, giving $s'_p = s_c + I_{xc}/(y'_c A) = 0.800 + 0.6912/(4.84145 \times 2.160) = 0.800 + 0.06610 = 0.8661\,\text{m}$, and $s_c < s'_p < s_p$ ($0.800 < 0.8661 < 0.9264$) as expected because uniform pressurization pulls the center of pressure closer to the centroid.

Exercise 6

Configuration Summary: *Vertical rectangular floodgate AB of height $L = 1.20\,\text{m}$ ($z \in [0, L]$) and width $b = 2.00\,\text{m}$, pivoting on a horizontal shaft O at height $d = 0.50\,\text{m}$ above the floor, with a downstream sill at B ($z = 0$) preventing OB from swinging*

outward while allowing automatic tipping when OA swings outward. Open reservoir of water depth $H \geq L$.

1. **For $H = L = 1.20\,\text{m}$, determine the resultant hydrostatic force on the gate, the height z_p of the center of pressure above the floor, and the horizontal reaction R_B exerted by the sill at B. Verify that the gate remains closed.** ★★☆☆☆

Method ***Vertical Plane Gate Formulas ($F_R = \gamma_w h_c A$, $h_p = h_c + I_{xc}/(h_c A)$) and Pivot Moment Balance***

> ***Why we do this (Way of Thinking):***
> For a vertical wall ($\theta = 90°$), slant distance y coincides with vertical depth $h = H - z$ measured downward from the free surface. Computing the depth h_p of the center of pressure and converting it to elevation above the floor via $z_p = H - h_p$ immediately reveals which way the water tries to rotate the gate about pivot O ($z_O = d = 0.50\,\text{m}$): whenever $z_p < d$, $\vec{F}_R$ acts *below* O, pressing the bottom edge B against the downstream sill and keeping the gate closed ($R_B > 0$).

Execution:

Step 1 — Gate area, centroidal inertia, and resultant force at $H = L = 1.20\,\text{m}$:

$$A = bL = (2.00\,\text{m})(1.20\,\text{m}) = 2.40\,\text{m}^2, \qquad I_{xc} = \frac{bL^3}{12} = \frac{(2.00\,\text{m})(1.20\,\text{m})^3}{12} = \frac{3.456}{12} = 0.288\,\text{m}^4.$$

At $H = L = 1.20\,\text{m}$, the gate centroid sits at elevation $z_c = L/2 = 0.60\,\text{m}$, so its depth below the surface is $h_c = H - z_c = 1.20 - 0.60 = 0.60\,\text{m}$:

$$F_R = \gamma_w h_c A = (9810\,\text{N m}^{-3})(0.60\,\text{m})(2.40\,\text{m}^2) = 14\,126.4\,\text{N} = 14.126\,\text{kN}.$$

Step 2 — Elevation z_p of the center of pressure above the floor:

$$h_p = h_c + \frac{I_{xc}}{h_c A} = 0.60\,\text{m} + \frac{0.288\,\text{m}^4}{(0.60\,\text{m})(2.40\,\text{m}^2)} = 0.60 + \frac{0.288}{1.44} = 0.60 + 0.20 = 0.80\,\text{m} \quad \left(= \frac{2}{3}L\right).$$

Converting depth h_p below $z = H = 1.20\,\text{m}$ to height z_p above the floor $z = 0$:

$$z_p = H - h_p = 1.20 - 0.80 = 0.40\,\text{m} \quad \left(= \frac{L}{3}\right).$$

Step 3 — Closure verification and sill reaction R_B: Because $z_p = 0.40\,\text{m} < d = 0.50\,\text{m}$, the rightward hydrostatic resultant $\vec{F}_R$ acts $\Delta z = d - z_p = 0.10\,\text{m}$ **below** pivot O, producing a counterclockwise moment that presses edge B rightward against the sill (so **the gate remains closed**). Taking moments about pivot O ($z_O = d = 0.50\,\text{m}$), where the sill exerts a leftward reaction R_B at $z_B = 0$ (lever arm $d = 0.50\,\text{m}$):

$$R_B d - F_R(d - z_p) = 0 \implies R_B = \frac{d - z_p}{d} F_R = \frac{0.50 - 0.40}{0.50}(14\,126.4\,\text{N}) = 0.20 \times 14\,126.4 = 2825.3\,\text{N}.$$

$$F_R = 14.13\,\text{kN}, \qquad z_p = 0.40\,\text{m} < d\ \text{(gate closed)}, \qquad R_B = 2.825\,\text{kN}$$

> ► **Common Pitfall:**
> Confusing depth $h_p = 0.80\,\text{m}$ (measured downward from the free surface) with elevation $z_p = 0.40\,\text{m}$ (measured upward from the floor) reverses the sign of the moment arm relative to pivot O ($z_O = 0.50\,\text{m}$) and falsely predicts that the gate opens at $H = 1.20\,\text{m}$.

2. **For $H > L$, express the height $z_p(H)$ of the center of pressure above the floor in terms of H and L, and**

determine the critical water depth H_c above which the gate automatically tips open. ★★★☆☆

Method ***Parametric Center-of-Pressure Locus $z_p(H)$ and Zero-Reaction Tipping Criterion***

> ***Why we do this (Way of Thinking):***
> As water rises above the top of the gate ($H > L$), the gate area $A = bL$ and centroid elevation $z_c = L/2 = 0.60\,\text{m}$ remain fixed, while the centroid depth $h_c(H) = H - L/2$ increases. Consequently, the relative pressure gradient across the gate becomes more nearly uniform, causing $z_p(H)$ to migrate upward from $L/3 = 0.40\,\text{m}$ toward the centroid $L/2 = 0.60\,\text{m}$. The gate tips open automatically at the exact threshold where $z_p(H_c)$ crosses the pivot height $d = 0.50\,\text{m}$ (so $R_B \to 0^+$).

Execution:

Step 1 — Analytic expression for $z_p(H)$: For any $H \geq L$, the gate centroid sits at depth $h_c = H - \dfrac{L}{2}$, and $\dfrac{I_{xc}}{A} = \dfrac{bL^3/12}{bL} = \dfrac{L^2}{12}$. Therefore:

$$h_p(H) = h_c + \frac{I_{xc}}{h_c A} = \left(H - \frac{L}{2}\right) + \frac{L^2}{12\left(H - \dfrac{L}{2}\right)}.$$

Subtracting $h_p(H)$ from the total water depth H cancels H outside the fraction:

$$z_p(H) = H - h_p(H) = H - \left(H - \frac{L}{2}\right) - \frac{L^2}{12\left(H - \dfrac{L}{2}\right)} = \frac{L}{2} - \frac{L^2}{12\left(H - \dfrac{L}{2}\right)}.$$

Step 2 — Critical tipping depth H_c: Tipping occurs as soon as $z_p(H) > d$ (releasing contact at the sill, $R_B = 0$). Setting $z_p(H_c) = d$:

$$\begin{aligned}
\frac{L}{2} - \frac{L^2}{12\left(H_c - \dfrac{L}{2}\right)} &= d \\
\frac{L^2}{12\left(H_c - \dfrac{L}{2}\right)} &= \frac{L}{2} - d \\
H_c - \frac{L}{2} = \frac{L^2}{12\left(\dfrac{L}{2} - d\right)} &\implies H_c = \frac{L}{2} + \frac{L^2}{12\left(\dfrac{L}{2} - d\right)}.
\end{aligned}$$

Substituting $L = 1.20\,\text{m}$ ($L/2 = 0.60\,\text{m}$) and $d = 0.50\,\text{m}$:

$$H_c = 0.60\,\text{m} + \frac{(1.20\,\text{m})^2}{12(0.60 - 0.50)\,\text{m}} = 0.60 + \frac{1.44}{1.20} = 0.60 + 1.20 = 1.80\,\text{m}.$$

$$\boxed{z_p(H) = \frac{L}{2} - \frac{L^2}{12\left(H - \frac{L}{2}\right)}, \qquad H_c = 1.80\,\text{m}}$$

> **✓ Sanity Check:**
> At $H = L$, $z_p(L) = \frac{L}{2} - \frac{L^2}{12(L/2)} = \frac{L}{2} - \frac{L}{6} = \frac{L}{3} = 0.40\,\text{m}$ (recovering Question 1); at $H_c = 1.80\,\text{m}$, $z_p(1.80) = 0.60 - 1.44/(12 \times 1.20) = 0.60 - 0.10 = 0.50\,\text{m} = d$.

3. If the pivot shaft O were placed at $d' = 0.65\,\text{m}$ above the floor instead of $0.50\,\text{m}$, determine whether the gate could ever tip open for any finite water depth H. Justify your answer. ★★★☆☆

Method ***Asymptotic Upper Bound of the Center of Pressure***

> ***Why we do this (Way of Thinking):***
> Because hydrostatic pressure always increases with depth, the lower half of the rectangular gate ($z \in [0, L/2]$) always experiences higher pressure than the upper half ($z \in [L/2, L]$). Hence the center of pressure $z_p(H)$ always lies strictly below the geometric centroid $z_c = L/2 = 0.60\,\text{m}$, approaching $L/2$ from below only as $H \to +\infty$. Placing the pivot at $d' = 0.65\,\text{m} > L/2$ puts O permanently above $z_p(H)$.

Execution: For every finite depth $H \geq L = 1.20\,\text{m}$, we have $H - L/2 > 0$, so the second term in $z_p(H)$ is strictly positive:

$$\frac{L^2}{12\left(H - \frac{L}{2}\right)} > 0 \implies z_p(H) = \frac{L}{2} - \frac{L^2}{12\left(H - \frac{L}{2}\right)} < \frac{L}{2} = 0.60\,\text{m}.$$

Moreover, $\lim_{H\to+\infty} z_p(H) = \dfrac{L}{2} = 0.60\,\text{m}$. Since $d' = 0.65\,\text{m} > 0.60\,\text{m}$, it follows that for **all** $H \geq L$:

$$z_p(H) < 0.60\,\text{m} < d' = 0.65\,\text{m} \implies d' - z_p(H) > 0.05\,\text{m} > 0.$$

Thus $\vec{F}_R$ always acts strictly below the shaft O, maintaining a positive closing moment $F_R(d' - z_p(H)) > 0$ against the bottom sill.

> No: $z_p(H) < \dfrac{L}{2} = 0.60\,\text{m} < d' = 0.65\,\text{m}$ for all $H \geq L$; the gate never tips open.

> ▸ **Common Pitfall:**
> Blindly plugging $d' = 0.65\,\text{m}$ into the algebraic formula $H_c = \frac{L}{2} + \frac{L^2}{12(L/2 - d')}$ yields $H_c = 0.60 + 1.44/(-0.60) = -1.80\,\text{m}$, which is unphysical because $H \geq L = 1.20\,\text{m}$ is required for full gate submergence.

Exercise 7

Configuration Summary: *Open water reservoir of total depth $H = 2.50\,\text{m}$ above floor $z = 0$. Vertical retaining wall at $x = 0$ for $z \in [R, H]$; overhanging quarter-circular cylindrical gate BA of radius $R = 1.50\,\text{m}$ and width $b = 2.00\,\text{m}$ between $B(0, R)$ (hinge) and $A(R, 0)$ (floor), with center of curvature $O'(R, R)$ on the dry atmospheric side. Horizontal leftward stop force F_A at A.*

(a) Fluid control volume V_{fillet} **(b) Gate BA FBD (line of action through O')**



1. Determine the magnitude F_H of the horizontal component of the hydrostatic force on gate BA and the

depth h_p of its line of action below the free surface. ★★☆☆☆

Method *Vertical Projection Theorem for Submerged Curved Surfaces*

> ***Why we do this (Way of Thinking):***
> Projecting the curved gate BA onto a vertical plane ($x = 0$) produces a submerged vertical rectangle of width $b = 2.00\,\text{m}$ and height $R = 1.50\,\text{m}$ extending from $z = 0$ (depth $H = 2.50\,\text{m}$) up to $z = R = 1.50\,\text{m}$ (depth $H - R = 1.00\,\text{m}$). Horizontal force equilibrium of the fluid fillet V_{fillet} under the arc requires F_H (directed rightward, $+x$) to equal the hydrostatic force acting on this vertical rectangular projection $A_v = bR$.

Execution: The vertical projection A_v is a rectangle of width $b = 2.00\,\text{m}$ and height $R = 1.50\,\text{m}$:

$$A_v = bR = (2.00\,\text{m})(1.50\,\text{m}) = 3.00\,\text{m}^2,$$
$$I_{xc} = \frac{bR^3}{12} = \frac{(2.00\,\text{m})(1.50\,\text{m})^3}{12} = \frac{2.00 \times 3.375}{12} = 0.5625\,\text{m}^4.$$

Its top edge sits at depth $H - R = 2.50 - 1.50 = 1.00\,\text{m}$ and its bottom edge at depth $H = 2.50\,\text{m}$, so its centroid depth is:

$$h_c = (H - R) + \frac{R}{2} = 1.00 + 0.75 = 1.75\,\text{m}.$$

Computing F_H (acting horizontally to the right) and its depth h_p below the free surface:

$$F_H = \gamma_w h_c A_v = (9810\,\text{N m}^{-3})(1.75\,\text{m})(3.00\,\text{m}^2) = 51\,502.5\,\text{N} = 51.503\,\text{kN},$$
$$h_p = h_c + \frac{I_{xc}}{h_c A_v} = 1.75\,\text{m} + \frac{0.5625\,\text{m}^4}{(1.75\,\text{m})(3.00\,\text{m}^2)} = 1.75 + \frac{0.5625}{5.25} = 1.75 + \frac{3}{28} \approx 1.8571\,\text{m}.$$

$$F_H = 51.50\,\text{kN (rightward)}, \qquad h_p = \frac{26}{14}\,\text{m} \approx 1.857\,\text{m}$$

> ✓ **Sanity Check:**
> The elevation of F_H above the floor is $z_p = H - h_p = 2.50 - 1.8571 = 0.6429\,\text{m}$, which lies slightly below the projection midpoint $R/2 = 0.75\,\text{m}$, as required for a vertical rectangle under hydrostatic loading.

2. Determine the magnitude and vertical direction (upward or downward) of the vertical component F_V of the hydrostatic force on gate BA, then deduce the magnitude of the resultant force F_R and its angle with the horizontal. ★★★★☆

Method *Fluid Control-Volume Vertical Equilibrium (Virtual Overlying Water Column)*

> ***Why we do this (Way of Thinking):***
> Inspecting the orientation of the fluid domain relative to the circular arc BA: because the center of curvature $O'(R, R)$ lies on the dry external side, the water fills the overhanging pocket *underneath* gate BA ($0 \le x \le R$). Consequently, at every point on arc BA, the inward normal from the water into the gate points **upward and rightward** toward $O'(R, R)$, so F_V acts **vertically upward** ($+z$). Isolating the real fluid fillet V_{fillet} bounded by $x = 0$, the floor $z = 0$, and arc BA proves that F_V equals the weight of the *virtual* water column extending from arc BA all the way up to the free surface plane $z = H$.

Execution:

Step 1 — Vertical force balance on the fluid fillet V_{fillet} ($x \in [0, R]$, $z \in [0, R]$): The real water fillet beneath arc BA has volume equal to the $R \times R$ square minus the quarter-circle of radius R:

$$V_{\text{fillet}} = b\left(R^2 - \frac{\pi R^2}{4}\right).$$

Along its bottom face on the floor ($z = 0$, area $A_h = bR$), the floor exerts an upward gauge force $F_{\text{floor}} = p_{\text{floor,gage}} A_h = (\gamma_w H)(bR)$. Meanwhile, gravity pulls the fillet downward with weight $W_{\text{fillet}} = \gamma_w V_{\text{fillet}}$, and gate BA pushes downward on the fluid with $-F_V$ (by action–reaction, the fluid pushes **upward** on the gate with $+F_V$). Vertical equilibrium of V_{fillet} ($\sum F_z = 0$) gives:

$$F_{\text{floor}} - W_{\text{fillet}} - F_V = 0 \implies F_V = \gamma_w bHR - \gamma_w b\left(R^2 - \frac{\pi R^2}{4}\right)$$
$$= \gamma_w b\left[R(H-R) + \frac{\pi R^2}{4}\right] = \gamma_w V_{\text{virt}},$$

which is precisely the weight of the virtual water column V_{virt} above arc BA up to $z = H$ (consisting of the upper rectangle $bR(H-R)$ plus the quarter-cylinder $b\,\pi R^2/4$):

$$V_{\text{virt}} = (2.00\,\text{m})\left[(1.50\,\text{m})(2.50-1.50)\,\text{m} + \frac{\pi(1.50\,\text{m})^2}{4}\right] = 2.00\,(1.50 + 1.767146) = 6.534\,29\,\text{m}^3,$$
$$F_V = \gamma_w V_{\text{virt}} = (9810\,\text{N m}^{-3})(6.534\,29\,\text{m}^3) = 64\,101.4\,\text{N} = 64.101\,\text{kN} \quad (\textbf{vertically upward}).$$

Step 2 — Resultant hydrostatic force magnitude F_R and angle α above the horizontal:

$$F_R = \sqrt{F_H^2 + F_V^2} = \sqrt{(51.5025)^2 + (64.1014)^2}\,\text{kN} = \sqrt{2652.51 + 4108.99}\,\text{kN} = 82.228\,\text{kN},$$
$$\alpha = \arctan\left(\frac{F_V}{F_H}\right) = \arctan\left(\frac{64.1014}{51.5025}\right) = \arctan(1.24463) = 51.22^\circ \quad (\text{above } +x).$$

$F_V = 64.10\,\text{kN}$ (upward), $F_R = 82.23\,\text{kN}$, $\alpha = 51.22^\circ$ above horizontal

> ► **Common Pitfall:**
> Computing only the weight of the real fillet $\gamma_w bR^2(1 - \pi/4)$ instead of including the upward floor pressure force $\gamma_w HbR$ (or equivalently the full virtual column up to $z = H$) misses both the correct magnitude and the upward direction of F_V.

3. Determine the horizontal force F_A required at A to hold the gate closed. ★★★★☆

Method *Concurrency of Circular-Arc Pressure Normals at the Center of Curvature $O'(R, R)$*

> ***Why we do this (Way of Thinking):***
> Every elementary hydrostatic pressure force $d\vec{F}_p = p\,dS\,\vec{n}$ acts normal to the circular cylinder BA, so every single force line passes radially through the center of curvature $O'(R, R)$ (Fig. b above). Therefore, the resultant hydrostatic force $\vec{F}_R = (F_H,\ F_V)$ can be sliding-transmitted directly to point $O'(R, R)$ with zero couple. Comparing the coordinates of $O'(R, R)$ relative to hinge $B(0, R)$ shows that both points lie at the **exact same elevation** $z = R$: at O', the horizontal component $\vec{F}_H$ passes straight through hinge B (zero lever arm), while the upward component $\vec{F}_V$ acts at horizontal lever arm R to the right of B.

Execution: Translating $\vec{F}_R = F_H\vec{i} + F_V\vec{k}$ along its radial line of action to the center of curvature $O'(R, R)$ and taking moments about the frictionless hinge $B(0, R)$ ($\sum M_B = 0$, counterclockwise positive):

- $\vec{F}_H$ at $O'(R, R)$ acts along the line $z = R$ passing directly through $B(0, R) \implies M_B(\vec{F}_H \text{ at } O') = 0$;
- $\vec{F}_V$ at $O'(R, R)$ acts vertically upward $(+z)$ along $x = R$ (lever arm R right of B) $\implies M_B(\vec{F}_V \text{ at } O') = +F_V R$;
- Stop force $\vec{F}_A$ at $A(R, 0)$ acts horizontally leftward $(-x)$ along $z = 0$ (lever arm R below B) $\implies M_B(\vec{F}_A) = -F_A R$.

Rotational equilibrium of the massless gate about hinge B yields immediately:

$$F_V R - F_A R = 0 \implies F_A = F_V = 64\,101.4\,\text{N} = 64.101\,\text{kN}.$$

$$F_A = F_V = 64.10\,\text{kN}$$

✓ Sanity Check:
Direct component-moment verification without shifting to O': $\vec{F}_H$ acts at elevation $z_p = H - h_p = 2.50 - \frac{26}{14} = \frac{9}{14}$ m (distance $R - z_p = \frac{12}{14}$ m below B), while $\vec{F}_V$ splits into the upper rectangle weight $F_{V1} = \gamma_w b R(H-R) = 29\,430.0$ N at $x_1 = R/2 = 0.75$ m and the quarter-circle weight $F_{V2} = \gamma_w b \frac{\pi R^2}{4} = 34\,671.4$ N at $x_2 = R - \frac{4R}{3\pi} = 0.863\,38$ m. Summing moments about B:

$$M_{B,\text{hydro}} = (51\,502.5)\left(\frac{12}{14}\right) + (29\,430.0)(0.75) + (34\,671.4)(0.86338)$$
$$= 44\,145.0 + 22\,072.5 + 29\,934.6 = 96\,152.1\,\text{N m},$$

and dividing by $R = 1.50$ m gives $F_A = 96\,152.1/1.50 = 64\,101.4$ N, matching F_V to six significant digits.

Exercise 8

Configuration Summary: *Open two-layer tank: lower water layer ($\rho_w = 1000\,\text{kg m}^{-3}$) of variable depth H topped by oil ($\rho_o = 800\,\text{kg m}^{-3}$, constant thickness $h_o = 1.00$ m). Vertical cylinder ($S = 0.55\,\text{m}^2$, $H_s = 0.80$ m, $\rho_s = 900\,\text{kg m}^{-3}$) tethered by a vertical cord ($L_c = 1.00$ m) from its lower center to the midpoint of edge B of a massless horizontal rectangular drain gate OB ($A_v = 0.020\,\text{m}^2$, hinged along O on the floor $z = 0$, resting on a recessed lip at B, air at p_{atm} underneath).*

1. **While the cord is slack ($T = 0$) and the cylinder floats completely submerged across the oil–water interface, determine the equilibrium height x_0 of the cylinder immersed in the water layer, and deduce the water depth H_0 at which the cord first becomes taut.** ★★★☆☆

Method ***Two-Fluid Archimedes Buoyancy Balance and Cord-Tautness Kinematic Condition***

Why we do this (Way of Thinking):
Notice the density ordering $\rho_o = 800\,\text{kg m}^{-3} < \rho_s = 900\,\text{kg m}^{-3} < \rho_w = 1000\,\text{kg m}^{-3}$: the solid cylinder is heavier than oil (so it sinks through the oil surface) yet lighter than water (so it floats stably across the oil–water interface). By Archimedes' principle applied to a body straddling two immiscible static fluids, the total upward buoyancy force equals the sum of the weights of the displaced water volume Sx_0 and the displaced oil volume $S(H_s - x_0)$. As H rises, the floating cylinder rides upward with the interface ($z_{\text{bot}} = H - x_0$) until z_{bot} reaches the cord length $L_c = 1.00$ m.

Execution:

Step 1 — Equilibrium water-immersion height x_0 for $T = 0$: With total cylinder volume $V_s = SH_s$, water-immersed volume $V_w = Sx_0$, and oil-immersed volume $V_o = S(H_s - x_0)$, vertical force equilibrium of the free-floating cylinder ($F_A - W_s = 0$) gives:

$$\rho_w g(Sx_0) + \rho_o g S(H_s - x_0) = \rho_s g(SH_s)$$
$$(\rho_w - \rho_o)x_0 + \rho_o H_s = \rho_s H_s$$
$$x_0 = \frac{\rho_s - \rho_o}{\rho_w - \rho_o} H_s = \frac{900 - 800}{1000 - 800}(0.80\,\text{m}) = \frac{100}{200} \times 0.80 = 0.40\,\text{m}.$$

(Check full submergence inside the oil layer: the upper portion in oil has height $H_s - x_0 = 0.80 - 0.40 = 0.40\,\text{m} < h_o = 1.00\,\text{m}$, confirming the cylinder top stays 0.60 m below the free oil surface.)

Step 2 — Water depth H_0 at onset of cord tension: With the oil–water interface at elevation $z = H$, the cylinder's bottom face floats at $z_{\text{bot}} = H - x_0$. The inextensible cord anchored at $z = 0$ first becomes taut ($z_{\text{bot}} = L_c$) at water depth H_0:

$$H_0 - x_0 = L_c \implies H_0 = L_c + x_0 = 1.00 + 0.40 = 1.40\,\text{m}.$$

$$x_0 = 0.40\,\text{m}, \qquad H_0 = 1.40\,\text{m}$$

✓ **Sanity Check:**
Since $\rho_s = 900\,\text{kg m}^{-3}$ is the exact arithmetic mean of $\rho_o = 800\,\text{kg m}^{-3}$ and $\rho_w = 1000\,\text{kg m}^{-3}$, the cylinder must straddle the interface symmetrically (50% in water, 50% in oil: $x_0 = H_s/2 = 0.40\,\text{m}$).

2. For $H \in [H_0,\ L_c + H_s]$ with the drain gate still closed, express the cord tension $T(H)$ in terms of ρ_w, ρ_o, S, g, H, and H_0. ★★★☆☆

Method *Tethered Two-Layer Buoyancy Increment Analysis*

Why we do this (Way of Thinking):
Once H exceeds $H_0 = 1.40\,\text{m}$ while the gate remains closed on its floor lip, the taut inextensible cord locks the cylinder's bottom face at elevation $z_{\text{bot}} = L_c = 1.00\,\text{m}$ and its top face at $z_{\text{top}} = L_c + H_s = 1.80\,\text{m}$. For every extra increase $H - H_0$ of the interface above H_0 (up to $H = L_c + H_s = 1.80\,\text{m}$, where water reaches the cylinder's top face), a slice of height $H - H_0$ and cross-section S previously surrounded by oil (ρ_o) becomes surrounded by denser water (ρ_w), generating an upward net buoyancy excess balanced directly by cord tension $T(H)$.

Execution: For $H \in [H_0,\ L_c + H_s] = [1.40,\ 1.80]\,\text{m}$, the cylinder bottom sits at $z = L_c$ and the interface sits at $z = H \in [L_c + x_0,\ L_c + H_s]$, so the water-immersed height is $x(H) = H - L_c$ and the oil-immersed height is $H_s - x(H)$ (with cylinder top at $L_c + H_s \le H + h_o$, fully submerged). Vertical equilibrium of the cylinder ($F_A(H) - W_s - T(H) = 0$) gives:

$$T(H) = \big[\rho_w x(H) + \rho_o\big(H_s - x(H)\big)\big]Sg - \rho_s H_s Sg.$$

Sub **Substitute $\rho_s H_s = \rho_w x_0 + \rho_o (H_s - x_0)$ from Question 1 and $x(H) - x_0 = (H - L_c) - (H_0 - L_c) = H - H_0$:**

$$\begin{aligned} T(H) &= \big[\rho_w x(H) + \rho_o H_s - \rho_o x(H) - \rho_w x_0 - \rho_o H_s + \rho_o x_0\big]Sg \\ &= (\rho_w - \rho_o)\big(x(H) - x_0\big)Sg \\ &= (\rho_w - \rho_o)\,(H - H_0)\,Sg. \end{aligned}$$

$$\boxed{T(H) = (\rho_w - \rho_o)\,Sg\,(H - H_0) \qquad (H_0 \le H \le L_c + H_s)}$$

► **Common Pitfall:**
Writing $T(H) = \rho_w Sg(H - H_0)$ instead of $(\rho_w - \rho_o)Sg(H - H_0)$ overestimates the cord tension by a factor of $\rho_w/(\rho_w - \rho_o) = 5$: as the interface rises along the tethered cylinder, water replaces *oil* (density gain $\rho_w - \rho_o = 200\,\text{kg m}^{-3}$), not air.

3. Determine the critical water depth H_c at which the drain gate OB automatically opens, and compute the cord tension at that instant. ★★★★☆

Method *Coupled Buoyancy–Hydrostatic Moment Threshold on a Horizontal Gate*

Why we do this (Way of Thinking):
Because the drain gate OB lies horizontally on the floor $z = 0$, the gauge hydrostatic pressure across its entire upper face A_v is uniform: $p_{\text{floor,gage}}(H) = \rho_o g h_o + \rho_w g H$. Therefore, the downward net hydrostatic force $F_p(H) = p_{\text{floor,gage}}(H) A_v$ acts at the geometric centroid of the rectangle OB, located at distance $w/2$ from hinge O (where $w = OB$). Meanwhile, the cord pulls vertically upward at the opposite edge B with lever arm w—twice the lever arm of $F_p(H)$—so the gate lifts off the recessed lip ($R_{\text{lip}} = 0$) as soon as $T(H_c) = \frac{1}{2} F_p(H_c)$.

Execution:

Step 1 — Downward hydrostatic force $F_p(H)$ and moment threshold about hinge O: With atmospheric pressure acting on both the top oil free surface and the underside of gate OB, the net downward force on the horizontal gate of area A_v and length $w = OB$ is:

$$F_p(H) = p_{\text{floor,gage}}(H)\, A_v = (\rho_o h_o + \rho_w H)\, g\, A_v, \qquad \text{acting at } \frac{w}{2} \text{ from } O.$$

At the instant of opening ($H = H_c$, lip contact reaction $R_{\text{lip}} = 0$), moment equilibrium of the massless gate about hinge O ($\sum M_O = 0$) gives:

$$T(H_c)\, w - F_p(H_c)\, \frac{w}{2} = 0 \implies T(H_c) = \frac{1}{2} F_p(H_c) = \frac{1}{2}(\rho_o h_o + \rho_w H_c)\, g\, A_v.$$

Step 2 — Solving for H_c and verifying the regime $H_c \in [H_0,\, L_c + H_s]$: Equating $T(H_c) = (\rho_w - \rho_o)Sg(H_c - H_0)$ to $\frac{1}{2}(\rho_o h_o + \rho_w H_c)gA_v$ and dividing both sides by g:

$$(\rho_w - \rho_o)S(H_c - H_0) = \frac{1}{2}\rho_o h_o A_v + \frac{1}{2}\rho_w A_v H_c$$
$$\left[(\rho_w - \rho_o)S - \tfrac{1}{2}\rho_w A_v\right] H_c = (\rho_w - \rho_o)SH_0 + \tfrac{1}{2}\rho_o h_o A_v.$$

Evaluating each coefficient numerically with $\rho_w - \rho_o = 200\,\text{kg m}^{-3}$, $S = 0.55\,\text{m}^2$, $H_0 = 1.40\,\text{m}$, $h_o = 1.00\,\text{m}$, and $A_v = 0.020\,\text{m}^2$:

$$(\rho_w - \rho_o)S = 200 \times 0.55 = 110.0\,\text{kg m}^{-1},$$
$$\tfrac{1}{2}\rho_w A_v = 0.5 \times 1000 \times 0.020 = 10.0\,\text{kg m}^{-1},$$
$$\tfrac{1}{2}\rho_o h_o A_v = 0.5 \times 800 \times 1.00 \times 0.020 = 8.00\,\text{kg}.$$

Substituting into the linear equation for H_c:

$$(110.0 - 10.0)\, H_c = (110.0)(1.40) + 8.00 = 154.0 + 8.00 = 162.0 \implies H_c = \frac{162.0}{100.0} = 1.62\,\text{m}.$$

Check regime validity: $H_0 = 1.40\,\text{m} \le H_c = 1.62\,\text{m} \le L_c + H_s = 1.80\,\text{m}$, confirming that the cylinder still straddles the oil–water interface at opening.

Step 3 — Cord tension $T(H_c)$ at the opening instant:

$$T(H_c) = (\rho_w - \rho_o)Sg(H_c - H_0) = (110.0\,\text{kg m}^{-1})(9.81\,\text{m s}^{-2})(1.62 - 1.40)\,\text{m}$$
$$= 110.0 \times 9.81 \times 0.22 = 237.402\,\text{N}.$$

$$\boxed{H_c = 1.62\,\text{m}, \qquad T(H_c) = 237.4\,\text{N}}$$

> ✓ **Sanity Check:**
> Independent check via the gate moment side: $F_p(H_c) = (800 \times 1.00 + 1000 \times 1.62)(9.81)(0.020) = 2420 \times 0.1962 = 474.804\,\text{N}$, so $\frac{1}{2}F_p(H_c) = 237.402\,\text{N} = T(H_c)$ identically.

4. If the cylinder cross-sectional area is reduced to $S' = 0.25\,\text{m}^2$ (all other parameters remaining unchanged), determine whether the drain gate ever opens for any water depth $H > 0$. Justify quantitatively.
★★★★★

Method *Piecewise Buoyancy Saturation Analysis Across All Depth Regimes ($H \gtrless L_c + H_s$)*

> ***Why we do this (Way of Thinking):***
> Substituting $S' = 0.25\,\text{m}^2$ directly into the linear equation of Question 3 yields a formal root $H^* > 1.80\,\text{m}$ that violates the geometric domain of validity of the two-layer buoyancy law. Specifically, the linear tension growth $T'(H) = (\rho_w - \rho_o)S'g(H - H_0)$ terminates at $H = L_c + H_s = 1.80\,\text{m}$, the exact depth where the oil–water interface reaches the **top face** of the tethered cylinder. For all $H \ge 1.80\,\text{m}$, the cylinder is 100% immersed in water alone, so its buoyancy and cord tension saturate at a constant ceiling $T'_{\max}$, whereas the downward hydrostatic load $\frac{1}{2}F_p(H)$ on the floor gate continues growing linearly with H.

Execution: Because $x_0 = \dfrac{\rho_s - \rho_o}{\rho_w - \rho_o} H_s = 0.40\,\text{m}$ and $H_0 = L_c + x_0 = 1.40\,\text{m}$ do not depend on cross-sectional area, we analyze all three depth intervals for $S' = 0.25\,\text{m}^2$:

Case (a): $0 < H \le H_0 = 1.40\,\text{m}$ **(slack cord):** $T'(H) = 0 < \frac{1}{2}F_p(H)$, so the gate remains closed.

Case (b): $1.40\,\text{m} < H \le L_c + H_s = 1.80\,\text{m}$ **(cylinder straddling interface):** Here $T'(H) = (\rho_w - \rho_o)S'g(H - 1.40)$ grows linearly with slope $(\rho_w - \rho_o)S'g = 200 \times 0.25 \times 9.81 = 490.5\,\text{N m}^{-1}$, reaching its maximum at $H = 1.80\,\text{m}$:

$$\begin{aligned} T'(1.80\,\text{m}) &= (200\,\text{kg m}^{-3})(0.25\,\text{m}^2)(9.81\,\text{m s}^{-2})(1.80 - 1.40)\,\text{m} \\ &= 50 \times 9.81 \times 0.40 = 196.20\,\text{N}. \end{aligned}$$

Compare against the required opening tension $\frac{1}{2}F_p(H)$ across $[1.40,\ 1.80]$ m:

$$\begin{aligned} \tfrac{1}{2}F_p(1.40) &= \tfrac{1}{2}(800 \times 1.00 + 1000 \times 1.40)(9.81)(0.020) = 215.82\,\text{N} > T'(1.40) = 0, \\ \tfrac{1}{2}F_p(1.80) &= \tfrac{1}{2}(800 \times 1.00 + 1000 \times 1.80)(9.81)(0.020) = 255.06\,\text{N} > T'(1.80) = 196.20\,\text{N}. \end{aligned}$$

Because both $T'(H)$ and $\frac{1}{2}F_p(H)$ are linear on $[1.40,\ 1.80]$ m and $T'(H) < \frac{1}{2}F_p(H)$ at both endpoints, $T'(H) < \frac{1}{2}F_p(H)$ everywhere on $[1.40,\ 1.80]$ m.

Case (c): $H > 1.80\,\text{m}$ **(cylinder completely submerged inside the water layer):** For all $H \ge 1.80\,\text{m}$, the entire cylinder volume $V_s' = S'H_s$ lies below the oil–water interface inside pure water (ρ_w), so cord tension remains permanently capped at:

$$T'(H) = T'_{\max} = (\rho_w - \rho_s)H_s S' g = (1000 - 900)(0.80)(0.25)(9.81) = 196.20\,\text{N} = \text{const},$$

whereas the required opening tension grows strictly beyond 255.06 N:

$$\frac{1}{2}F_p(H) > \frac{1}{2}F_p(1.80\,\text{m}) = 255.06\,\text{N} > 196.20\,\text{N} = T'(H) \qquad (\forall\, H > 1.80\,\text{m}).$$

Consequently, $T'(H) < \frac{1}{2}F_p(H)$ for every water depth $H > 0$.

No: $T'_{\max} = 196.20\,\text{N} < \frac{1}{2}F_p(1.80\,\text{m}) = 255.06\,\text{N}$; the drain gate never opens for any $H > 0$.

► **Common Pitfall:**
Substituting $S' = 0.25\,\text{m}^2$ into the unclipped Question 3 formula gives $(50 - 10)H^* = 50(1.40) + 8 = 78 \implies H^* = 1.95\,\text{m}$. Because $1.95\,\text{m} > L_c + H_s = 1.80\,\text{m}$, that root lies outside the domain of validity of the two-layer formula and is physically unattainable.