Set 01 ”basics”

Exercise 1:

An insurance company classifies its policyholders into three categories: low risk, medium risk, and high risk individuals.
Its statistics indicate that the probability that a person is involved in an accident over a one-year period is respectively:
0.05, 0.15, and 0.30.
It is estimated that:

- 20% of the population is low risk,
- 50% is medium risk, and
- 30% is high risk.

a) What is the probability that a policyholder has at least one accident during a given year?
b) If a certain policyholder did not have an accident last year, what is the probability that they belong to the low-risk category?

Exercise 2:

An individual in a certain population may suffer from unilateral or bilateral visual impairment. It is assumed that, in the general population, the probability of being affected in the right eye is equal to that of being affected in the left eye, and that these two events are independent. Let this probability be denoted by p.

1) Calculate the probability of having:
   a) at least one eye affected,
   b) no eye affected,
   c) exactly one eye affected,
   d) both eyes affected.
2) What do you notice from question (1)? Justify your answer.
3) Now consider an individual who has a visual impairment. Calculate the probability that they have:
   - a unilateral impairment,
   - a bilateral impairment.
4) Let:
   - V: the event “the individual has a visual impairment”,
   - D: the event “the individual is affected in the right eye”,
   - G: the event “the individual is affected in the left eye”.

   Compute $P(D \mid V)$ and $P(G \mid V)$, and deduce that D and G are not independent given V.

Exercise 3:

A player plays several rounds of a video game. The probability of winning the first game is 0.1.

- If he wins a game, the probability of winning the next game is 0.8.
- If he loses a game, the probability of winning the next game is 0.6.

Let:

- G_n: "the player wins the n-th game"
- $p_n = P(G_n)$

1) Calculate p_2.
2) The player won the second game; calculate the probability that he lost the first game.
3) Calculate the probability that the player wins **at least one** of the first three games.
4) Show that, for all $n \in \mathbb{N}^*$:
$$p_{n+1} = \frac{1}{5}p_n + \frac{3}{5}$$
5) Assume that the sequence (p_n) converges to l; find l.

Exercise 4:

A country is made up of two tribes, A and B, which are trying to make peace. There are three times as many inhabitants in A as in B.
The inhabitants of A are 60% in favor of peace, 16% are opposed, and 24% have no opinion. On the other hand, the inhabitants of B are 68% in favor of war, 12% are opposed, and 20% have no opinion.
You meet a randomly chosen inhabitant of this country and ask for their opinion.
a) Calculate the probability that they have no opinion.
b) If they answer that they are in favor of peace, what is the probability that they belong to tribe A?
c) If they answer that they are in favor of war, what is the probability that they belong to tribe B?

Exercise 5:

13. A box contains 7 white balls and 9 black balls.
a) Two balls are drawn from the box. What is the probability that they are of the same color?
b) If two balls of the same color are drawn, what is the probability that they are white?
c) If two white balls have already been drawn, what is the probability that a third ball drawn is also white?
d) If three balls are drawn from this box, what is the probability that at least one of them is black?
Now the box is emptied and refilled with:

- x white balls, and
- $x + 2$ black balls (i.e., two more black balls than white balls).

a) Show that the probability $p(x)$ of drawing two balls of the same color is:

$$p(x) = \frac{x^2 + x + 1}{2x^2 + 3x + 1}$$

b) What is the total number of balls that must be placed in the box so that this probability $p(x)$ is as small as possible?

Exercise 6:

Consider the following game: a player first rolls a fair (unbiased) die. Then, they draw a token from an urn chosen according to the result of the die.

- Urn U_1 is chosen if the result is 1, 2, or 3.
- Urn U_2 is chosen if the result is 4 or 5.
- Urn U_3 is chosen if the result is 6.

The urns contain the following tokens:

- Urn U_1: two red tokens and three blue tokens;
- Urn U_2: two blue tokens and four green tokens;
- Urn U_3: one green token and one red token.

1) What is the probability of obtaining a green token using this process?
2) Given that a green token is obtained, what is the probability that it came from urn U_2?
3) Given that a blue token is obtained, what is the probability that the die showed 3?
4) What is the probability of not obtaining a green token, given that the die result is 3 or 6?
5) Are the events "choosing urn U_3" and "obtaining a green token" independent? Justify your answer.

Exercise 7:

We consider two urns:

- Urn A contains 3 red balls and 2 black balls.
- Urn B contains 2 red balls and 5 black balls.

One of the urns is chosen at random. A ball is drawn from that urn and then placed into the other urn. Then, a ball is drawn from the second urn.

1) Draw a probability tree diagram.
2) Calculate the probability that the first ball drawn is red.
3) If the first ball drawn is red, what is the probability that it came from urn A?
4) Calculate the probability that the two balls drawn are of the same color.

■ HOMEWORK

Exercise 8:

We consider two urns, U_1 and U_2:

- Urn U_1 contains 4 red balls and 2 white balls.
- Urn U_2 contains 2 red balls and 4 white balls.

An urn is chosen at random, and successive draws are made from it **with replacement**.
Let:

- R_i: the event "obtaining a red ball on the i-th draw"
- U_1: the event "the draws are made with replacement from urn U_1"
- U_2: the event "the draws are made with replacement from urn U_2"

1) Calculate the probability of obtaining a red ball on the first draw.
2) Calculate the probability of obtaining a red ball on the third draw, given that red balls were obtained in the first two draws.
3) Let P_n be the probability that the draws were made from urn U_1, given that red balls were obtained in the first n draws.

   Find P_n, and determine its limit as $n \to \infty$.

   (Hint: use Bayes' formula.)