Exercise Series

Continuous Random Variables

Exercise 1 Verifying a Density and Using the CDF

Let X be a continuous random variable with density:

$$f_X(x) = 2x, \quad \text{if } 0 < x < 1, \qquad f_X(x) = 0 \text{ otherwise.}$$

1. Verify that f_X is a valid probability density function.
2. Compute $P\left(X > \frac{1}{2}\right)$.
3. Determine the cumulative distribution function (cdf) $F_X(x)$.
4. Compute $P\left(\frac{1}{3} < X < \frac{2}{3}\right)$ using F_X.

Exercise 2 Finding a Constant and a Probability

Let X be a continuous random variable with density:

$$f_X(x) = C(1 - x^2), \quad \text{if } 0 < x < 1, \qquad f_X(x) = 0 \text{ otherwise.}$$

1. Determine the value of C.
2. Compute $P\left(\frac{1}{4} < X < \frac{3}{4}\right)$.

Exercise 3 Expected Value and Variance

Let X be a continuous random variable with density:

$$f_X(x) = 2(1 - x), \quad \text{if } 0 < x < 1, \qquad f_X(x) = 0 \text{ otherwise.}$$

1. Verify that f_X is a valid probability density function.
2. Compute $E(X)$.
3. Compute $E(X^2)$.
4. Deduce $\mathrm{Var}(X)$ and the standard deviation $\sigma(X)$.

Exercise 4 A Piecewise Density

Let X be a continuous random variable with density:

$$f_X(x) = \begin{cases} Cx, & 0 \leq x \leq 2 \\ C(4 - x), & 2 < x \leq 4 \\ 0, & \text{otherwise.} \end{cases}$$

1. Determine C.
2. Determine $F_X(x)$.
3. Compute $P(1 < X < 3)$.
4. Compute $E(X)$.

Exercise 5 A Density with Chebyshev's Inequality

Consider a random variable X with density:

$$f(x) = \begin{cases} cx(1-x), & 0 \leq x \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Evaluate the constant c so that f is a probability density.

(b) Determine the cumulative distribution function F of X.

(c) Compute $P\left(\frac{1}{2} < X < \frac{3}{4}\right)$.

(d) Determine the expected value and variance of X.

(e) Find a lower bound for $P\left(\frac{1}{2} < X < \frac{3}{4}\right)$ using Chebyshev's inequality.

Exercise 6 Finding Constants, Expected Value, Variance & Variable Transformation

Let X be a continuous random variable with the following probability density function (pdf):

$$f(x) = \begin{cases} 3Cx^2, & 0 \leq x \leq C \\ 0, & \text{otherwise.} \end{cases}$$

1. Determine the value of the constant C. Then calculate the expected value $E(X)$ and the variance $\text{Var}(X)$.
2. Find the cumulative distribution function (cdf) $F(x)$ of X.
3. Compute the following probabilities:
   - $P\left(0 < X \leq \frac{1}{2}\right)$
   - $P(X = 1)$
4. Let Y be a new random variable defined as:

$$Y = \begin{cases} 1, & 0 < X \leq \frac{1}{2} \\ 0, & \text{otherwise.} \end{cases}$$

   Determine the probability distribution of Y.

Exercise 7 Theoretical Derivation of the Uniform Distribution

Let X be a continuous random variable defined over the interval $[a, b]$ with a constant probability density function:

$$f(x) = \begin{cases} k, & a \leq x \leq b \\ 0, & \text{otherwise,} \end{cases}$$

where k is a positive real constant ($k \in \mathbb{R}^+$).

1. Determine the value of k so that $f(x)$ is a valid probability density function, and show that the expected value is $E(X) = \frac{a+b}{2}$.
2. Determine the cumulative distribution function (cdf) $F(x)$ of X.

<u>Exercise 8</u> Advanced PDF with Absolute Value & Transformation

Let X be a continuous random variable with the probability density function defined on $\mathbb{R}$ by:

$$f(x) = ke^{-|x|}$$

where k is a positive real constant ($k \in \mathbb{R}^+$).

1. Determine the value of k so that f is a valid density function.
2. Determine its cumulative distribution function (cdf) $F(x)$.
3. Let $Y = X^2$. Calculate the cumulative distribution function $F_Y(y)$ of Y.

<u>Exercise 9</u> The Exponential Distribution

Let $\lambda > 0$ be fixed. A random variable X is said to follow an exponential distribution with parameter λ if X has density

$$f(x) = \lambda e^{-\lambda x}\mathbf{1}_{\{x\geq 0\}},$$

denoted $X \sim \mathcal{E}(\lambda)$.

1. Sketch f. Verify that f is indeed a probability density.
2. Compute and sketch the cumulative distribution function F.
3. Compute the expected value and variance of X.
4. The lifetime T (in years) of a television follows the density $f(t) = \dfrac{1}{8}e^{-t/8}\mathbf{1}_{\{t\geq 0\}}$.
   4.1. What is the average lifetime of such a television? And its standard deviation?
   4.2. Compute the probability that your television has a lifetime greater than 8 years.

<u>Exercise 10</u> Markov's Inequality and the Median

Suppose we only know that the average annual rainfall in a country is 800 millimeters. What is:

(1) the maximum probability that it rains at least 1200 millimeters in a year?

(2) the largest possible value for the median of the annual total rainfall?

Exercise 11 Normal Distribution

Let X be a random variable following a normal distribution with mean $m = 20$ and variance $\sigma^2 = 16$.

1. Determine σ.
2. Compute $P(X < 24)$.
3. Compute $P(16 < X < 28)$.
4. Compute $P(X > 12)$.

Use $Z = \dfrac{X - m}{\sigma}$, where Z follows the standard normal distribution $\mathcal{N}(0, 1)$.

Exercise 12 Normal Distribution — Inverse Problem

Let X be a random variable following a normal distribution $\mathcal{N}(50, 10^2)$.

1. Determine a such that $P(X < a) = 0.95$.
2. Determine b such that $P(X > b) = 0.10$.
3. Determine k such that $P(50 - k < X < 50 + k) = 0.95$.

Use the standard normal distribution table $\mathcal{N}(0, 1)$.

Exercise 13 Proof of the Standard Normal Transformation

Let X be a continuous random variable following a normal distribution with mean m and variance σ^2, denoted $X \sim \mathcal{N}(m, \sigma^2)$. Its probability density function is given by:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$$

Using the cumulative distribution function (cdf) approach, prove that the transformed random variable $Z = \dfrac{X - m}{\sigma}$ follows the standard normal distribution $\mathcal{N}(0, 1)$, whose density function is:

$$f_Z(z) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}z^2}$$

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