Exercise Series

Joint Random Variables

Course: Probability & Statistics
Level: Undergraduate
Essential Exercises

Exercise 1 — Discrete Random Vector on $\mathbb{N}^*$

Let X and Y be two discrete random variables taking values in $\mathbb{N}^* = \{1, 2, 3, \dots\}$. Suppose their joint distribution is defined as:

$$P(X = m, Y = n) = a \cdot 2^{-(m+n)}, \quad 1 \leq m \leq n,$$
$$P(X = m, Y = n) = 0 \quad \text{otherwise.}$$

1. Show that $a = \dfrac{3}{2}$.
2. Find the marginal distribution of X. (Hint: use the sum of a geometric series.)
3. Find the marginal distribution of Y.
4. Are X and Y independent? Justify your answer.

Exercise 2 — Hiring Exam Selection

During an international hiring exam, candidates are randomly assigned 4 problem-solving tasks from a total pool of 12 questions covering three different topics: 4 Advanced Mathematics questions, 5 Computer Science questions, and 3 Physics questions. Let X denote the number of selected questions related to Advanced Mathematics, and let Y denote the number of selected questions related to Computer Science.

1. Determine the joint probability distribution and the marginal probability distributions of the random vector (X, Y).
2. Compute the covariance $\mathrm{Cov}(X, Y)$.
3. Are X and Y independent? Justify your answer using two different methods.

Exercise 3

Let (X, Y) be a pair of continuous random variables whose joint probability density is

$$f(x, y) = \begin{cases} 4xy, & 0 \leq x \leq 1,\ 0 \leq y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

1. Determine the marginal probability density of X and that of Y.
2. Are X and Y independent? Justify your answer.
3. Calculate $E(X \mid Y)$.
4. Calculate $P(X > Y)$.

Exercise 4

Let (X, Y) be a pair of random variables whose joint probability density is

$$f(x,y) = \begin{cases} C\left(\dfrac{y}{2} + \dfrac{xy}{2}\right), & 0 < x < 2,\ 0 < y < 1, \\ 0, & \text{otherwise,} \end{cases}$$

where C is a real constant.

1. Determine the value of C.
2. Determine the marginal probability density of X, then that of Y.

Exercise 5

The random variables X and Y have the joint density

$$f(x,y) = \begin{cases} e^{-x^2 y}, & x \geq 1,\ y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

1. Calculate $P(X^2 Y > 1)$.
2. Calculate the marginal densities $f_X(x)$ and $f_Y(y)$.
3. Are X and Y independent?

Exercise 6 — The Two Friends Problem

Two friends A and B agree to meet after work between 6:00 p.m. and 7:00 p.m. Let X be the fraction of an hour between 6:00 p.m. and the arrival of A, and let Y be the fraction of an hour between 6:00 p.m. and the arrival of B. Assume that X and Y are independent and uniformly distributed on $[0, 1]$.

Let T be the waiting time of the first friend to arrive.

1. Express T in terms of X and Y.
2. Find the cumulative distribution function of T and deduce its probability density function.
3. Calculate the mathematical expectation of T and its second centered moment.
4. The friends agree that the first person to arrive will wait for the other for at most 30 minutes. What is the probability that the meeting takes place?

Exercise 7 — Best Predictor

The dependence between X and Y, representing the scores on two exams, is modeled by the joint density

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{756}\left(x^2 + xy\right), & 0 < x < 6,\ 0 < y < 6, \\ 0, & \text{otherwise.} \end{cases}$$

1. Find the marginal densities f_X and f_Y.
2. Find the conditional density $f_{Y|X}$ of Y given X.
3. Calculate Z, the best predictor of Y based on X, and its mathematical expectation.

Exercise 8 — Discrete Random Vector

Let X and Y be two discrete random variables with joint probability mass function given by:

$$P(X = i, Y = j) = c \cdot (i + j), \quad i \in \{1, 2, 3\},\ j \in \{1, 2\},$$

$$P(X = i, Y = j) = 0 \quad \text{otherwise.}$$

1. Find the value of the constant c.
2. Construct the joint probability distribution table of (X, Y).
3. Find the marginal distributions of X and Y, and compute their expectations $E(X)$ and $E(Y)$.
4. Compute $P(X \geq 2 \mid Y = 1)$.
5. Compute $\text{Cov}(X, Y)$ and determine whether X and Y are independent.

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