correction of set 01

Exercise 1:

Let the following events and probabilities be defined:

- A: "have an accident or more"
- L: "belong to the low-risk category"
- M: " " medium " "
- H: " " high " "

With given information:

$$P(L) = 0.2, \quad P(M) = 0.5, \quad P(H) = 0.3$$

$$P(A \mid L) = 0.05, \quad P(A \mid M) = 0.15, \quad P(A \mid H) = 0.3$$

a) $P(A)$?

$$P(A) = P(L) \cdot P(A \mid L) + P(M) \cdot P(A \mid M) + P(H) \cdot P(A \mid H)$$
$$= 0.05 \times 0.2 + 0.15 \times 0.5 + 0.3 \times 0.3 = 0.175$$

b) $P(L \mid \bar{A})$?

$$P(L \mid \bar{A}) = \frac{P(\bar{A} \mid L) \cdot P(L)}{P(\bar{A})} = \frac{1 - P(A \mid L)}{1 - P(A)} \cdot P(L) = \frac{(1 - 0.05) \times 0.2}{1 - 0.175} \approx 0.23$$

Exercise 2:

Let the following events be defined:

- R: "affected in the right eye" $\quad P(R) = P(L) = p$
- L: "affected in the left eye" $\quad P(R \cap L) = p^2$

a) $P(R \cup L) = P(R) + P(L) - P(R \cap L) = p + p - p^2 = 2p - p^2$

b) $P(\overline{R \cup L}) = 1 - P(R \cup L) = 1 - (2p - p^2) = (1 - p)^2$

c) $P(R \cap \bar{L}) \cup (\bar{R} \cap L)) = P(R \cap \bar{L}) + P(\bar{R} \cap L) = P(R) \times P(\bar{L}) + P(\bar{R}) \times P(L)$

$$= p(1 - p) + p(1 - p) = 2p(1 - p)$$

d) $P(R \cap L) = p^2$

2) Elementary partition.

Let V: "having a visual impairment".

$$P(V) = 2p - p^2, \quad P(U \mid V) = \frac{P(U \cap V)}{P(V)} = \frac{P(U)}{P(V)} \quad (U \subset V), \quad P(V) = 2p(1 - p)$$

$$= \frac{2p(1 - p)}{2p - p^2} = \frac{2(1 - p)}{2 - p}$$

$$P(B \mid V) = \frac{P(B)}{P(V)} = \frac{p}{2-p}$$

$$4.\ P(R \mid V) = \frac{P(R)}{P(V)} = \frac{p}{2-p} = P(L \mid V)$$

$$P(R \cap L \mid V) = \frac{p^2}{2-p}$$

Exercise 3:

Given:

$$P(G_1) = 0.1, \quad P(G_2 \mid G_1) = 0.8, \quad P(G_2 \mid \bar{G}_1) = 0.6$$

1) $P(G_2) = P(G_1) \times P(G_2 \mid G_1) + P(\bar{G}_1) \times P(G_2 \mid \bar{G}_1) = 0.1 \times 0.8 + 0.9 \times 0.6 = 0.62$

2) $P(\bar{G}_1 \mid G_2) = \dfrac{P(\bar{G}_1) \times P(G_2 \mid \bar{G}_1)}{P(G_2)} = \dfrac{0.9 \times 0.6}{0.62} = 0.871$

3) $P(G_3 \mid \bar{G}_2) = ?$

$$P(G_1 \cup G_2 \cup G_3) = 1 - P(\bar{G}_1 \cap \bar{G}_2 \cap \bar{G}_3) = 1 - P(\bar{G}_1) \times P(\bar{G}_2 \mid \bar{G}_1) \times P(\bar{G}_3 \mid \bar{G}_2 \cap \bar{G}_1)$$

$$= 1 - 0.9 \times 0.4 \times 0.4 = 0.856$$

4) $P(G_{n+1}) = P(G_n) \times P(G_{n+1} \mid G_n) + P(\bar{G}_n) \times P(G_{n+1} \mid \bar{G}_n)$

$$= 0.8\, p_n + 0.6 \times (1 - p_n) = 0.2\, p_n + 0.6 = \frac{1}{5} p_n + \frac{3}{5}$$

5) $n \to +\infty$:

$$\ell = \frac{1}{5} \times \ell + \frac{3}{5} \implies \frac{4}{5}\ell = \frac{3}{5} \implies \ell = \frac{3}{4}$$

Exercise 4:

Let:

- A: "belong to A", B: "belong to B"
- C: "in favour of peace", N: "is opposed of peace", V: "No opinion"

Given: $P(A) = 3\,P(B)$ and $P(A) + P(B) = 1 \implies P(A) = \dfrac{3}{4}$, $P(B) = \dfrac{1}{4}$.

$$P(C \mid A) = 0.6, \quad P(W \mid A) = 0.16, \quad P(N \mid A) = 0.24$$

$$P(C \mid B) = 0.12, \quad P(W \mid B) = 0.68, \quad P(N \mid A) = 0.2$$

1) $P(N) = P(A) \times P(N \mid A) + P(B) \times P(N \mid B) = 0.23$

2) $P(A \mid C) = \dfrac{P(A) \times P(C \mid A)}{P(C)} = 0.9375$

3) $P(B \mid W) = 0.5862$

Ex 7:

$$P(R_1 \mid A) = \frac{3}{5}, \quad P(R_2 \mid R_1, \cap A)$$

1) $P(R) = P(A) \times P(R \mid A) + P(B) \times P(R \mid B) = 0.442$

2) $P(A \mid R) = \dfrac{P(A) \times P(R \mid A)}{P(R)} = 0.677$

3) $P(R_1 \cap R_2) + P(N_1 \cap N_2) = 0.536$

$$= P(R_1 \cap R_2 \cap A) + P(R_1 \cap R_2 \cap B)$$

$$= P(A) \times P(R_1 \mid A) \times P(R_2 \mid R_1 \cap A) + \cdots$$

Exercise 5:

I)

Let W be the event "women", A the event "selected person".

1) $P(A) = \dfrac{A_7^2 + A_3^2}{A_{16}^2} = \dfrac{10}{40}$ (where CA denotes chosen arrangement)

2) $P(W \mid A) = P(W \cap A) \cdot \dfrac{P(W)}{P(A)} = \dfrac{A_7^2 \times A_3^2}{A_{16}^2} \times \dfrac{40}{10} = \dfrac{7}{10}$

3) $P(W' \mid \cdot) = \dfrac{P(A)}{5} \cdot \dfrac{1}{14}$

4) $P(B) = 1 - P(W_3), \quad P(W_3) = \dfrac{A_3^3}{A_{16}^3} = \dfrac{1}{16}$

$$P(B) = 1 - \frac{1}{16} = \frac{15}{16}$$

II)

$$1 - P(x) = \frac{A_x^2}{A_{2x+2}^2} + \frac{A_{x+2}^2}{A_{2x+2}^2} = \frac{\frac{x!}{(x-2)!}}{\frac{(2x+2)!}{(2x)!}} + \frac{\frac{(x+2)!}{x!}}{\frac{(2x+2)!}{(2x)!}}$$

$$= \frac{x(x-1)}{(2x+2)(2x+1)} + \frac{(x+2)(x+1)}{(2x+2)(2x+1)} = \frac{x^2 - x + x^2 + 3x + 2}{4x^2 + 6x + 2} = \frac{2x^2 + 2x + 2}{4x^2 + 6x + 2} = \frac{x^2 + x + 1}{2x^2 + 3x + 1}$$

We put $g(x) = \dfrac{x^2 + x + 1}{2x^2 + 3x + 1}$ and $g'(x) = \dfrac{x^2 - 2x - 2}{(2x^2 + 3x + 1)^2}$.

$g'(x) = 0 \implies x^2 - 2x - 2 = 0 \implies x \approx 1 + \sqrt{3} \approx 2.73.$

For $x = 2$: $P(2) = \dfrac{7}{15} \approx 0.4666$

For $x = 3$: $P(3) = \dfrac{13}{28} \approx 0.4642 \quad \implies x = 3.$

Exercise 5:

Ex B:

$$P(U_1) = \frac{3}{6} = \frac{1}{2}, \quad P(U_2) = \frac{2}{6} = \frac{1}{3}, \quad P(U_3) = \frac{1}{6}$$

1- $P(G) = P(G \mid U_1) \times P(U_1) + P(G \mid U_2) \times P(U_2) + P(G \mid U_3) \times P(U_3)$

$$P(G) = 0 \times \frac{1}{2} + \frac{4}{6} \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{6} = \frac{11}{36}$$

2- $P(U_2 \mid G) = \dfrac{P(U_2) \times P(G \mid U_2)}{P(G)} = \dfrac{8}{11}$

3- $P(\text{Die } 3 \mid B)$?

$$P(B) = \frac{3}{5} \times \frac{1}{2} + \frac{2}{6} \times \frac{1}{3} + 0 \times \frac{1}{6} = \frac{37}{90}, \quad P(\text{Die } 3) = \frac{1}{6}$$

$$P(B \cap \text{Die } 3) = \frac{1}{6} \times \left[\frac{3}{5}\right] = \frac{1}{10} \implies P(B \mid U_1) = P(B \mid \text{Die} = 3) = \frac{3}{5}$$

$$P(\text{Die } 3 \mid B) = \frac{1}{10} \times \frac{90}{37} = \frac{9}{37}$$

4- $P(\bar{G} \mid \text{die} = 3 \cup \text{die} = 6) = 1 - P(G \mid \text{die} = 3 \cup \text{die} = 6)$

$$P(\text{die} = 3 \cup \text{die} = 6) = P(\text{die} = 3) + P(\text{die} = 6) = \frac{2}{6}$$

$$P(G \cap (\text{Die} = 3 \cup \text{Die} = 6)) = P(G \cap \text{Die} = 3) + P(G \cap \text{Die} = 6)$$

$$= \frac{1}{12} + 0, \quad \text{since } P(G \mid \text{Die} = 6) = P(G \mid U_3) = \frac{1}{2} \times P(\text{die} = 6) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$$

$$\implies = \frac{3}{4}$$

5- $P(U_3) = \frac{1}{6}$, $P(G) = \frac{11}{36}$, $P(U_3 \cap G) = \frac{1}{12}$, not independent.