ELECTROSTATICS

Complete Solutions — Three Classical Problems

Electric Line Charge · Charged Disk · Electric Dipole

#	Exercise	Topic
10	Finite Line Charge	Field, potential, energy of a uniformly charged rod
11	Uniformly Charged Disk	Axial field, potential, infinite-plane limit
12	Electric Dipole	Point-dipole, field lines, dipole in external field

Exercise 10

Electric Field of a Finite Uniform Line Charge

Rod length $L = 6.00$ cm $= 0.06$ m, surface charge density $\lambda = 4$ pC/m $= 4\times10^{\blacksquare12}$ C/m, rod centred at origin along the x-axis.

① Total Charge Q on the Rod

Integrating the linear charge density over the full length:

$$Q = \int \lambda \, dx = \lambda \cdot L$$

$$Q = \lambda L = (4\times10^{\blacksquare12})(0.06) = 2.4\times10^{\blacksquare13}\ \mathrm{C} \approx 0.24\ \mathrm{pC}$$

② Electric Field E(y) at Arbitrary Point P on y-axis

Consider a small element dx at position x on the rod. Its distance to $P(0,y)$ is $r = \sqrt{x^2+y^2}$. The contribution to the field has magnitude:

$$dE = (1/4\pi\varepsilon\blacksquare) \cdot \lambda \, dx / (x^2+y^2)$$

By symmetry, x-components of opposite elements cancel. Only the y-component survives, with projection factor $y/\sqrt{x^2+y^2}$. Integrating from $-L/2$ to $+L/2$ using $\int dx/(x^2+y^2)^{(3/2)} = x / [y^2\sqrt{x^2+y^2}]$:

$$E(y) = (\lambda / 4\pi\varepsilon\blacksquare) \cdot L / [y \cdot \sqrt{y^2 + (L/2)^2}] \ \blacksquare$$

③ Electric Potential V(y)

Using $dV = (1/4\pi\varepsilon\blacksquare)(\lambda \, dx / \sqrt{x^2+y^2})$ and the given integral formula:

$$V(y) = (\lambda/4\pi\varepsilon\blacksquare)\ [\ln(x + \sqrt{x^2+y^2})] \text{ from } x=-L/2 \text{ to } x=+L/2$$

$$V(y) = (\lambda/4\pi\varepsilon\blacksquare) \cdot \ln[\, (L/2 + \sqrt{(L/2)^2+y^2}) / (-L/2 + \sqrt{(L/2)^2+y^2}) \,]$$

④ Electric Field via E = –dV/dy

Differentiating $V(y)$ with respect to y, using $d/dy\,[\ln(a+R)] = y/[R(a+R)]$ where $a = L/2$, $R = \sqrt{a^2+y^2}$:

$$dV/dy = (\lambda/4\pi\varepsilon\blacksquare) \cdot [y/(R(a+R)) - y/(R(R-a))] = (\lambda/4\pi\varepsilon\blacksquare) \cdot (-2ay) / (y^2R)$$

$$E(y) = -dV/dy = (\lambda/4\pi\varepsilon\blacksquare) \cdot L / [y \cdot \sqrt{y^2+(L/2)^2}] \ \checkmark \ \text{(identical to ②)}$$

⑤ Comparison

The results of ② (direct integration) and ④ (–grad V) are identical, confirming internal consistency of the solutions. ✓

⑥ Potential in the Limit y >> L

For $y >> L$: $\sqrt{(L/2)^2+y^2} \approx y(1 + L^2/8y^2)$. The argument of the logarithm becomes:

$$num/den \approx (1 + L/2y) / (1 - L/2y) \approx 1 + L/y \rightarrow \ln(1 + L/y) \approx L/y$$

$$V(y) \approx (\lambda/4\pi\varepsilon\blacksquare) \cdot (L/y) = (1/4\pi\varepsilon\blacksquare) \cdot Q/y \text{ [point-charge potential]}$$

⑦ Electric Field in the Limit y >> L

$$E(y) = -dV/dy \approx (1/4\pi\varepsilon\blacksquare) \cdot Q/y^2 \text{ [Coulomb field of a point charge Q]}$$

⑧ Comparison with a Point Charge

The expression $E = Q/(4\pi\varepsilon\blacksquare y^2)$ is exactly Coulomb's law for a point charge Q at distance y. This confirms that far from the rod ($y >> L$), the extended charge distribution appears as a point charge — a fundamental result of electrostatics.

⑨ Change in Potential Energy — Proton from y = 2L to y = 6L

With $L = 0.06$ m, $a = L/2 = 0.03$ m, $k = 8.988\times10\blacksquare$ N·m²/C²:

$$\text{At } y = 2L = 0.12 \text{ m: } \sqrt{a^2+y^2} = 0.12369 \text{ m}$$

$$V(2L) = k\lambda \cdot \ln(0.15369 / 0.09369) = k\lambda \cdot \ln(1.6405) = 0.01780 \text{ V}$$

$$\text{At } y = 6L = 0.36 \text{ m: } \sqrt{a^2+y^2} = 0.36124 \text{ m}$$

$$V(6L) = k\lambda \cdot \ln(0.39124 / 0.33124) = k\lambda \cdot \ln(1.1812) = 0.005979 \text{ V}$$

$$\Delta U = e \cdot [V(6L) - V(2L)] = (1.602\times10\blacksquare^{1}\blacksquare)(0.005979 - 0.01780)$$

$$\Delta U \approx -1.893\times10\blacksquare^{21} \text{ J}$$

The potential energy decreases — the proton moves in the direction of the electric force.

⑩ Proton Velocity at y = 6L (starting from rest at y = 2L)

By conservation of energy: $\frac{1}{2} m_p v^2 = -\Delta U = 1.893\times10\blacksquare^{21}$ J

$$v = \sqrt{(2 \times 1.893\times10\blacksquare^{21} / 1.673\times10\blacksquare^{2}\blacksquare)} = \sqrt{(2.263\times10\blacksquare)}$$

$$v \approx 1.505\times10^3 \text{ m/s} \approx 1.50 \text{ km/s}$$

Summary — Exercise 10

Step	Result
①	$Q = \lambda L = 0.24$ pC
②	$E(y) = \lambda L / [4\pi\varepsilon\blacksquare\, y \sqrt{y^2+(L/2)^2}]\ \blacksquare$
③	$V(y) = (\lambda/4\pi\varepsilon\blacksquare) \ln[(L/2+R)/(R-L/2)]$, $R=\sqrt{(L/2)^2+y^2}$

④⑤	$E = -dV/dy$ confirms ② ✓
⑥	$y >> L$: $V \approx Q/(4\pi\varepsilon_0 y)$ [point charge]
⑦⑧	$y >> L$: $E \approx Q/(4\pi\varepsilon_0 y^2)$ [Coulomb law]
⑨	$\Delta U = -1.893 \times 10^{-21}$ J
⑩	$v \approx 1.505 \times 10^3$ m/s

Exercise 11

Electric Field of a Uniformly Charged Disk

Disk radius $R = 2$ cm = 0.02 m, surface charge density $\sigma = 6\ \mu C/m^2 = 6\times10\blacksquare\blacksquare$ C/m^2. Point M on the z-axis at height z above the disk centre.

① Charge Element dq

Take an annular ring at radius r with radial width dr. Its area element is $dA = 2\pi r\, dr$. Hence:

$$dq = \sigma\, dA = 2\pi\sigma\, r\, dr$$

② Total Charge Q

$$Q = \sigma \int\blacksquare\blacksquare\, 2\pi r\, dr = 2\pi\sigma\, [r^2/2]\blacksquare\blacksquare = \pi\sigma R^2$$

$$Q = \pi\sigma R^2 = \pi \times (6\times10\blacksquare\blacksquare) \times (0.02)^2 \approx 7.54\ \text{nC}$$

③ Electric Field Contribution dE

Each ring is at distance $\blacksquare = \sqrt{(r^2+z^2)}$ from M. The field element magnitude:

$$dE = (1/4\pi\varepsilon\blacksquare) \cdot dq / (r^2+z^2) = \sigma\, r\, dr / [2\varepsilon\blacksquare\, (r^2+z^2)]$$

④ Symmetry Argument — Field Points in +z Direction

For every area element at (r, ϕ) there is a diametrically opposite element at $(r, \phi+\pi)$. Their radial (horizontal) field components are equal and opposite, so they cancel exactly. The z-components both point in $+z$ and add constructively. By full azimuthal (2π) symmetry, only the z-component survives. The projection factor is $\cos\theta = z/\sqrt{(r^2+z^2)}$.

$$E\blacksquare(M) = E_z(M)\ \blacksquare$$

⑤ Total Electric Field E(M)

$$E_z = (\sigma z/2\varepsilon\blacksquare) \int\blacksquare\blacksquare\, r\, dr / (r^2+z^2)^{(3/2)}$$

Using substitution $u = r^2+z^2$, $du = 2r\, dr$:

$$\int\blacksquare\blacksquare\, r\, dr / (r^2+z^2)^{(3/2)} = [-1/\sqrt{(r^2+z^2)}]\blacksquare\blacksquare = 1/z - 1/\sqrt{(R^2+z^2)}$$

$$E\blacksquare(z) = (\sigma/2\varepsilon\blacksquare)\, [1 - z/\sqrt{(R^2+z^2)}]\ \blacksquare$$

Numerical check at $z = R = 0.02$ m:

$$E = (6\times10\blacksquare\blacksquare)/(2\times8.854\times10\blacksquare^{12}) \times (1 - 1/\sqrt{2}) \approx 9.93\times10\blacksquare\ \text{V/m}$$

⑥ Electric Potential V(z)

V(z) = (σ/2ε■) ∫■■ r dr / √(r²+z²) = (σ/2ε■) [√(r²+z²)]■■

V(z) = (σ/2ε■) (√(R²+z²) – z)

Verification: –dV/dz = (σ/2ε■)(1 – z/√(R²+z²)) = E_z ✓

⑦ Infinite Plane Limit (R >> z)

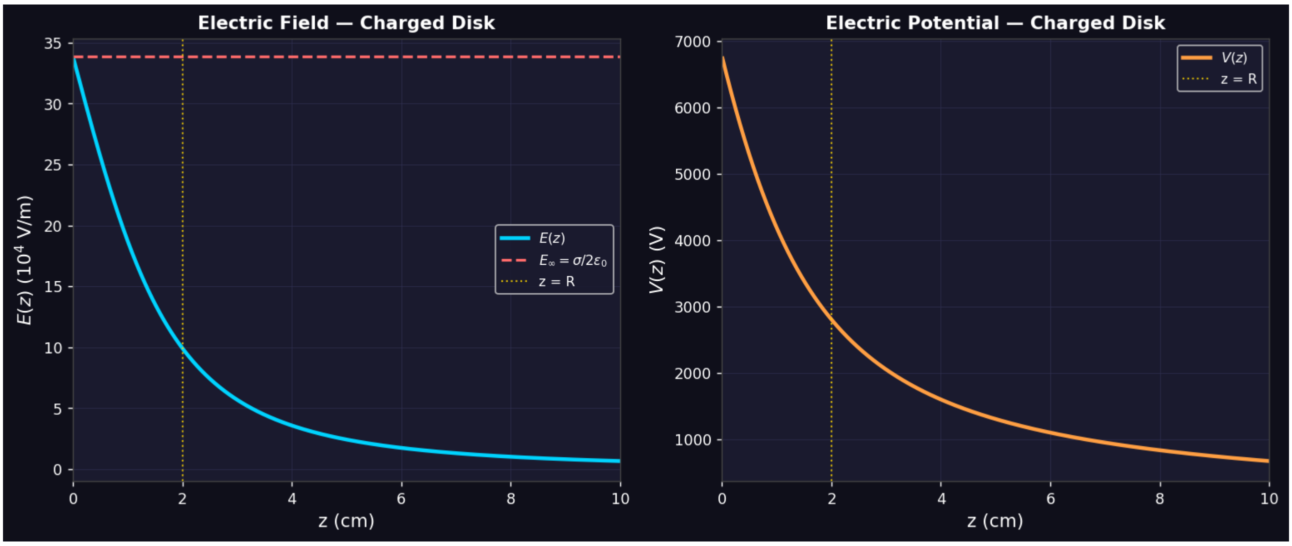
When R >> z: z/√(R²+z²) ≈ z/R → 0, so the term in brackets approaches 1:

E_∞ = σ/2ε■ = (6×10■■)/(2×8.854×10■¹²) ≈ 3.39×10■ V/m (uniform, z-independent)

This is the classic result for an infinite uniformly charged plane.

⑧ Plots of E(z) and V(z)

The plots below show E(z) (cyan) decreasing from σ/2ε■ toward zero, and V(z) (orange) decreasing monotonically from its maximum at the disk surface.



⑨ Numerical Values at Key Distances

z	z/R	E(z) (V/m)	V(z) (V)
0■	0	3.39×10■	6782
R = 2 cm	1	9.93×10■	4777
2R = 4 cm	2	4.43×10■	3502
5R = 10 cm	5	1.37×10■	2198
∞	∞	0	0

Exercise 12

Electric Dipole — Point-Dipole Approximation & External Field

Dipole: charges +q and –q separated by 2a, dipole moment $p = 2qa$. Polar coordinates (r, θ). Constant: $p/4\pi\varepsilon\blacksquare = 0.02\ \text{V·m}^2$.

Part A — Isolated Dipole

① Electric Potential at P by Superposition

The two charges are at ±a from the origin. Their distances to P are r± where $r_\pm^2 = r^2 + a^2 \blacksquare 2ar\cos\theta$. By superposition:

$$V(P) = (q/4\pi\varepsilon\blacksquare)\,[1/r\blacksquare - 1/r\blacksquare]$$

② Point-Dipole Limit (r >> a)

Using the Taylor expansion (given hints) to first order in a/r:

$$1/r\blacksquare \approx (1/r)(1 + (a/r)\cos\theta) \text{ and } 1/r\blacksquare \approx (1/r)(1 - (a/r)\cos\theta)$$

$$V = (q/4\pi\varepsilon\blacksquare)(1/r)[(1+(a/r)\cos\theta) - (1-(a/r)\cos\theta)] = (q/4\pi\varepsilon\blacksquare)(2a\cos\theta/r^2)$$

$$V(r,\theta) = (p/4\pi\varepsilon\blacksquare)\cdot\cos\theta / r^2 \quad [p = 2qa]$$

③ Electric Field of the Point Dipole

In polar coordinates, $E\blacksquare = -\nabla V$. The two components are:

$$E_r = -\partial V/\partial r = (p/4\pi\varepsilon\blacksquare)\cdot 2\cos\theta / r^3$$

$$E_\theta = -(1/r)\partial V/\partial\theta = (p/4\pi\varepsilon\blacksquare)\cdot \sin\theta / r^3$$

$$E\blacksquare(r,\theta) = (p / 4\pi\varepsilon\blacksquare r^3)\,(2\cos\theta\ r\blacksquare + \sin\theta\ \theta\blacksquare)$$

④ Field Line Equation — Integration

The field line condition $dr/(r\,d\theta) = E_r/E_\theta = 2\cos\theta/\sin\theta$ gives:

$$dr/r = 2\cot\theta\, d\theta$$

$$\ln r = 2\ln(\sin\theta) + C \rightarrow r = A\sin^2\theta$$

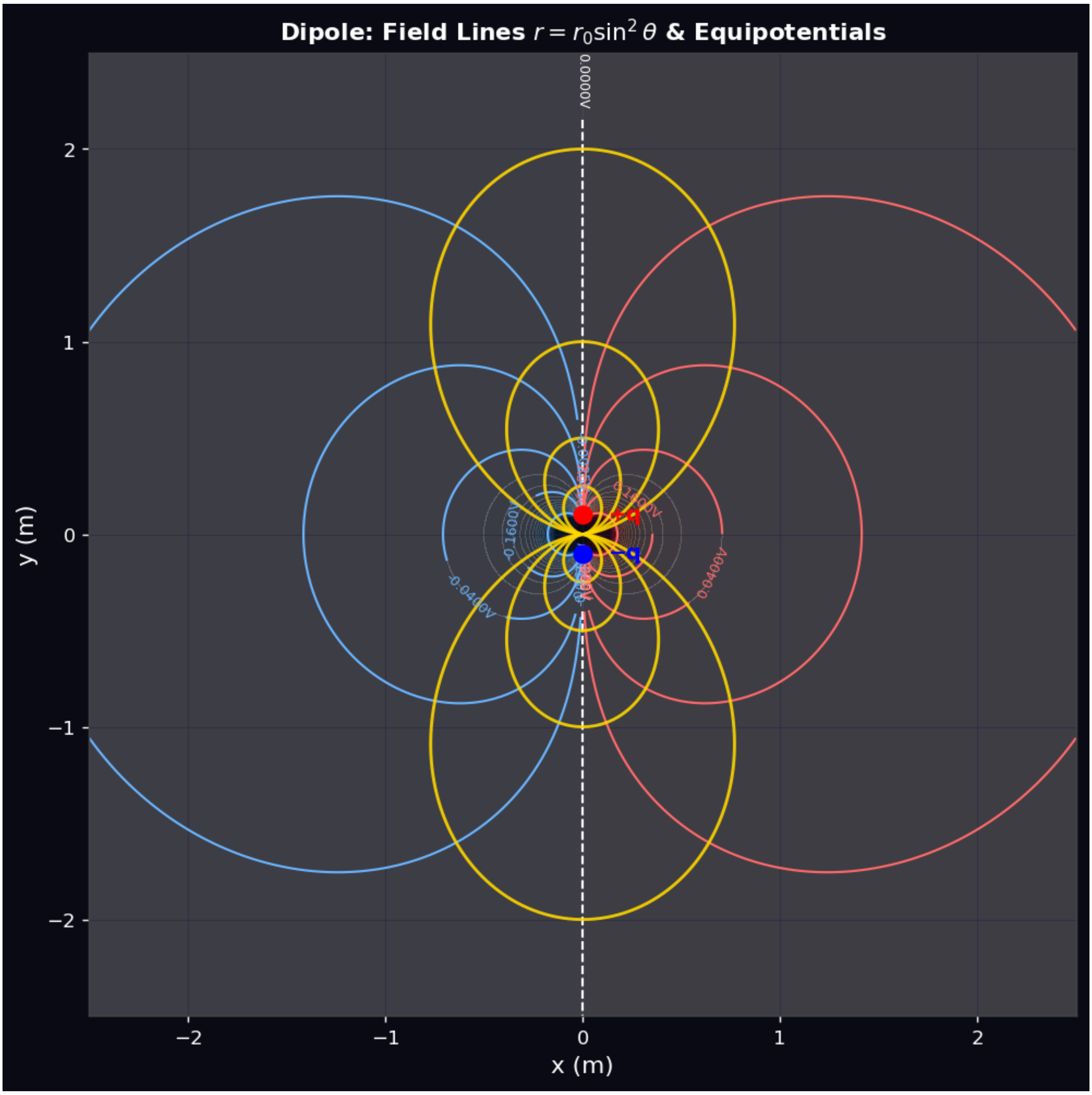
Boundary condition: at $\theta = \pi/2$, $r = r\blacksquare$ ■ $A = r\blacksquare$. Hence:

$$r = r\blacksquare \sin^2\theta$$

Each field line is characterised by its equatorial radius r■ (the distance from the dipole where the line crosses the perpendicular bisector).

⑤ Plots — Field Lines and Equipotentials

Gold curves: field lines r = r■ sin²θ for r■ = 0.25, 0.5, 1.0, 2.0 m. Red: positive equipotentials (V > 0). Blue: negative equipotentials (V < 0). Dashed white: V = 0 plane (θ = π/2).



Part B — Dipole in Uniform External Field E■ = E■ x■

① Total Potential (Dipole + Uniform Field)

The uniform field E■ along x contributes V_ext = –E■ x = –E■ r cosθ (zero at origin). Adding the dipole potential:

V(r,θ) = (p/4πε■r²) cosθ – E■ r cosθ = cosθ [p/(4πε■r²) – E■r]

② Equipotential Surface V = 0

V = 0 either when $\cos\theta = 0$ (the equatorial plane $\theta = \pi/2$), or when the bracket vanishes:

$$p/(4\pi\varepsilon\blacksquare r\blacksquare^2) = E\blacksquare r\blacksquare \rightarrow r\blacksquare^3 = p/(4\pi\varepsilon\blacksquare E\blacksquare)$$

$$\boxed{r\blacksquare = [p/(4\pi\varepsilon\blacksquare E\blacksquare)]^{(1/3)}}$$

With $p = (1/9)\times10\blacksquare^1\blacksquare$ C·m, $E\blacksquare = 10\blacksquare$ V/m, $k = 9\times10\blacksquare$ N·m²/C²:

$$r\blacksquare = [(10\blacksquare^1\blacksquare/9)(9\times10\blacksquare) / 10\blacksquare]^{(1/3)} = [10\blacksquare\blacksquare]^{(1/3)}$$

$$\boxed{r\blacksquare \approx 2.154\times10\blacksquare^2 \text{ m} \approx 2.15 \text{ cm}}$$

The V = 0 surface is a sphere of radius $r\blacksquare = 2.15$ cm centred on the dipole (plus the equatorial plane). Inside this sphere the dipole field dominates; outside, the uniform field dominates.

③ Electric Field on the V = 0 Equipotential

Compute $E\blacksquare = -\nabla V$, giving components:

$$E_r = \cos\theta\ [\,2p/(4\pi\varepsilon\blacksquare r^3) + E\blacksquare\,] \text{ and } E_\theta = \sin\theta\ [\,p/(4\pi\varepsilon\blacksquare r^3) - E\blacksquare\,]$$

On the sphere $r = r\blacksquare$ where $p/(4\pi\varepsilon\blacksquare r\blacksquare^3) = E\blacksquare$:

$$E_r|_{r\blacksquare} = \cos\theta\ (2E\blacksquare + E\blacksquare) = 3E\blacksquare \cos\theta$$

$$E_\theta|_{r\blacksquare} = \sin\theta\ (E\blacksquare - E\blacksquare) = 0$$

$$\boxed{E\blacksquare|_{V=0} = 3E\blacksquare \cos\theta\ r\blacksquare}$$

The field is purely radial on the equipotential sphere ($E_\theta = 0$ is consistent with V = const on that surface ✓) and has magnitude three times the applied field at the poles.

Summary — Exercise 12

Step	Result
A①	$V = (q/4\pi\varepsilon\blacksquare)(1/r\blacksquare - 1/r\blacksquare)$
A②	$V(r,\theta) = (p/4\pi\varepsilon\blacksquare) \cos\theta/r^2$
A③	$E\blacksquare = (p/4\pi\varepsilon\blacksquare r^3)(2\cos\theta\ r\blacksquare + \sin\theta\ \theta\blacksquare)$
A④	Field lines: $r = r\blacksquare \sin^2\theta$
B①	$V = \cos\theta[p/(4\pi\varepsilon\blacksquare r^2) - E\blacksquare r]$
B②	V=0 sphere: $r\blacksquare = [p/(4\pi\varepsilon\blacksquare E\blacksquare)]^{(1/3)} \approx 2.15$ cm
B③	$E\blacksquare|_{V=0} = 3E\blacksquare \cos\theta\ r\blacksquare$ ✓