Solutions and Summary of Laws for Selected Electrostatics Exercises

1 Solutions to Exercises

1.1 Exercise 1 – Origin of the Electric Charge

1 **Structure of the atom:** An atom consists of a small, dense, positively charged nucleus surrounded by a cloud of negatively charged electrons. The nucleus contains protons (positive) and neutrons (neutral).

2 **Characteristics:**

- Proton: mass $m_p \approx 1.6726 \times 10^{-27}\,\text{kg}$, charge $+e = +1.602 \times 10^{-19}\,\text{C}$.
- Neutron: mass $m_n \approx 1.6749 \times 10^{-27}\,\text{kg}$, charge 0.
- Electron: mass $m_e \approx 9.109 \times 10^{-31}\,\text{kg}$, charge $-e = -1.602 \times 10^{-19}\,\text{C}$.

3 **Total charge of ${}^{2}_{1}\text{X}$ (neutral atom):** The notation ${}^{2}_{1}\text{X}$ is ambiguous; assuming it represents a generic neutral atom with equal numbers of protons and electrons, the total charge is $0\,\text{C}$ because the positive and negative charges exactly balance.

4 **Atom that lost three electrons:** Net charge $= +3e = +3 \times 1.602 \times 10^{-19}\,\text{C} = +4.806 \times 10^{-19}\,\text{C}$. Symbol: X^{3+} (a cation).

5 **Atom that gained two electrons:** Net charge $= -2e = -2 \times 1.602 \times 10^{-19}\,\text{C} = -3.204 \times 10^{-19}\,\text{C}$. Symbol: X^{2-} (an anion).

1.2 Exercise 3 – Comparison between Newton‘s and Coulomb’s Laws

Part 1: Glass rod and silk cloth

(a) **Phenomenon:** Rubbing transfers electrons from the glass rod to the silk cloth because silk has a greater electron affinity. The rod becomes electron-deficient (positively charged) and the silk gains electrons (negatively charged).

(b) **Charge on glass rod:** $n = 30 \times 10^{10}$ electrons removed.

$$Q_{\text{rod}} = +ne = 30 \times 10^{10} \times 1.602 \times 10^{-19}\,\text{C} = 4.806 \times 10^{-8}\,\text{C} = 48.06\,\text{nC}.$$

(c) **Charge on silk:** By conservation of charge, $Q_{\text{silk}} = -Q_{\text{rod}} = -48.06\,\text{nC}$.

Part 2: Electrophorus and aluminum balls

(a) **Procedure and Principle:**

- A glass plate is rubbed with cat's fur, becoming positively charged.
- A metal disk with an insulating handle is placed on the charged plate. Charges separate inside the disk: negative charges are attracted to the lower surface, positive charges repelled to the upper surface.
- Grounding the disk (touching it with a finger) allows the repelled charges to escape. When the finger is removed and the disk lifted, it retains a net negative charge (opposite to the plate).
- Bringing this charged disk into contact with two initially neutral conducting balls shares the charge. After removal, both balls carry charge of the same sign and repel each other.

(b) **Free body diagram:** (Description) For each ball:

- Weight mg acting downward.
- Tension T along the string.
- Coulomb repulsive force F_e horizontal, away from the other ball.

The ball is in equilibrium; the string makes an angle $\alpha = 10^\circ$ with the vertical.

(c) **Charge on each ball:** Radius of ball $r = 0.5\,\text{cm} = 0.005\,\text{m}$, density $\rho = 2.7\,\text{g/cm}^3 = 2700\,\text{kg/m}^3$.

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(0.005)^3 = 5.236\times10^{-7}\,\text{m}^3, \quad m = \rho V = 2700\times5.236\times10^{-7} = 1.414\times10^{-3}\,\text{kg}.$$

$$mg = 1.414 \times 10^{-3} \times 9.8 = 0.013\,85\,\text{N}.$$

String length $L = 0.1\,\text{m}$, $\alpha = 10^\circ$. Distance between centers: $d = 2L\sin\alpha = 0.2\sin 10^\circ = 0.034\,73\,\text{m}$. From equilibrium: $T\cos\alpha = mg$, $T\sin\alpha = F_e = \dfrac{kq^2}{d^2}$.

$$F_e = mg\tan\alpha = 0.01385 \times \tan 10^\circ = 0.002\,442\,\text{N}.$$

$$q = \sqrt{\frac{F_e d^2}{k}} = \sqrt{\frac{0.002442 \times (0.03473)^2}{9 \times 10^9}} \approx 1.81 \times 10^{-8}\,\text{C} = 18.1\,\text{nC}.$$

Each ball carries $q \approx \pm 18.1\,\text{nC}$ (the sign is the same on both balls, determined by the disk's charge; here negative).

(d) **New charge on the metal disk:** Without the capacitance of the disk, we cannot calculate the exact leftover charge. In many textbook treatments it is assumed that all the charge is transferred to the balls, leaving the disk neutral. Under that assumption, the new charge is 0. If the disk retains charge, it would be $Q_{\text{initial}} - 2q$, where Q_{initial} is unknown. We state the result as **zero** under the complete-transfer assumption.

(e) **Gravitational force between the balls:**

$$F_g = \frac{Gm^2}{d^2} = \frac{6.67 \times 10^{-11} \times (1.414 \times 10^{-3})^2}{(0.03473)^2} \approx 1.11 \times 10^{-13}\,\text{N}.$$

1.3 Exercise 6 – Superposition Principle

1. **Free body diagram:** On q_4 (at $(0, y)$) act:
   - Repulsive forces $\vec{F}_{14}$ and $\vec{F}_{24}$ from q_1 and q_2 (both $+6\,\mathrm{nC}$). Their horizontal components cancel by symmetry; their vertical components add upward for $y > 0$.
   - Attractive force $\vec{F}_{34}$ from $q_3 = -6\,\mathrm{nC}$ directed downward (toward the origin).
2. **Position y_0 for zero net force:** q_1 at $(-0.1, 0)$, q_2 at $(0.1, 0)$, q_3 at $(0, 0)$. For $q_4 = +10\,\mathrm{nC}$ at $(0, y)$:

   $$F_{1y} = F_{2y} = \frac{kq_1q_4}{(0.1^2 + y^2)} \frac{y}{\sqrt{0.1^2 + y^2}},$$

   $$F_{3y} = -\frac{k|q_3|q_4}{y^2} \quad \text{(downward)}.$$

   Setting net vertical force to zero:

   $$2 \times \frac{k(6 \times 10^{-9})(10 \times 10^{-9})y}{(0.01 + y^2)^{3/2}} = \frac{k(6 \times 10^{-9})(10 \times 10^{-9})}{y^2}.$$

   Cancel common factors:

   $$\frac{2y}{(0.01 + y^2)^{3/2}} = \frac{1}{y^2} \implies 2y^3 = (0.01 + y^2)^{3/2}.$$

   Square both sides: $4y^6 = (0.01 + y^2)^3$. Let $u = y^2$:

   $$4u^3 = (0.01 + u)^3 \implies 4^{1/3}u = 0.01 + u.$$

   $4^{1/3} \approx 1.5874$, so $0.5874\,u = 0.01 \Rightarrow u \approx 0.017\,02\,\mathrm{m}^2$, $y_0 \approx 0.1305\,\mathrm{m} = 13.05\,\mathrm{cm}$. The position is $y = \pm 13.05\,\mathrm{cm}$.
3. **Net electric field at that point:** Since the force on q_4 is zero, the electric field $\vec{E} = \vec{F}/q_4 = \mathbf{0}$.
4. **Electric potential at $P(0, y_0)$:**

   $$r_1 = r_2 = \sqrt{0.1^2 + y_0^2} = \sqrt{0.01 + 0.01702} \approx 0.1644\,\mathrm{m}, \quad r_3 = y_0 \approx 0.1305\,\mathrm{m}.$$

   $$V = k\left(\frac{q_1}{r_1} + \frac{q_2}{r_2} + \frac{q_3}{r_3}\right) = 9 \times 10^9 \left[\frac{2 \times 6 \times 10^{-9}}{0.1644} - \frac{6 \times 10^{-9}}{0.1305}\right]$$
   $$\approx 9 \times 10^9 \times (7.300 \times 10^{-8} - 4.598 \times 10^{-8}) \approx 2.43 \times 10^2\,\mathrm{V} = 243\,\mathrm{V}.$$
5. **Work done by external agent moving q_4 from infinity to P:**

   $$W_{\text{ext}} = q_4(V_P - V_\infty) = (10 \times 10^{-9})(243 - 0) = 2.43 \times 10^{-6}\,\mathrm{J} = 2.43\,\mu\mathrm{J}.$$

6. **Electrical potential energy of q_4 in the fields of the other three:**

$$U = q_4 V_P = 2.43 \times 10^{-6}\,\text{J}.$$

7. **Total energy of the four–charge system (assembly energy):** Sum over all pairs $U = \sum_{i<j} \dfrac{kq_iq_j}{r_{ij}}$:

$$\begin{aligned}
U_{12} &= 9\times 10^9 \frac{(6\times 10^{-9})^2}{0.2} = 1.62\times 10^{-6}\,\text{J},\\
U_{13} &= 9\times 10^9 \frac{(6\times 10^{-9})(-6\times 10^{-9})}{0.1} = -3.24\times 10^{-6}\,\text{J},\\
U_{23} &= -3.24\times 10^{-6}\,\text{J},\\
U_{14} &= 9\times 10^9 \frac{(6\times 10^{-9})(10\times 10^{-9})}{0.1644} \approx 3.28\times 10^{-6}\,\text{J},\\
U_{24} &\approx 3.28\times 10^{-6}\,\text{J},\\
U_{34} &= 9\times 10^9 \frac{(-6\times 10^{-9})(10\times 10^{-9})}{0.1305} \approx -4.14\times 10^{-6}\,\text{J}.
\end{aligned}$$

$$U_{\text{total}} \approx (1.62 - 3.24 - 3.24 + 3.28 + 3.28 - 4.14)\times 10^{-6} = -2.44\times 10^{-6}\,\text{J}.$$

1.4 Exercise 10 – Electric Field of a Finite Uniform Line of Charge

Given: $L = 6.00\,\text{cm} = 0.06\,\text{m}$, $\lambda = 4\,\text{pC/m} = 4\times 10^{-12}\,\text{C/m}$.

1 Total charge: $Q = \lambda L = 4\times 10^{-12}\times 0.06 = 2.4\times 10^{-13}\,\text{C} = 0.24\,\text{pC}$.

2 Electric field at $(0, y)$ (perpendicular bisector):

$$\vec{E}(y) = \frac{kQ}{y\sqrt{y^2 + (L/2)^2}}\,\hat{\jmath}.$$

Derivation: $dE_y = \frac{k\lambda dx\, y}{(x^2+y^2)^{3/2}}$, integrate from $-L/2$ to $L/2 \Rightarrow E_y = \frac{2k\lambda}{y}\frac{L/2}{\sqrt{(L/2)^2+y^2}} = \frac{kQ}{y\sqrt{y^2+(L/2)^2}}$.

3 Electric potential (potential zero at infinity):

$$\begin{aligned}
V(y) &= \int_{-L/2}^{L/2} \frac{k\lambda\,dx}{\sqrt{x^2+y^2}} = k\lambda \ln\left(\frac{L/2 + \sqrt{(L/2)^2+y^2}}{-L/2 + \sqrt{(L/2)^2+y^2}}\right)\\
&= 2k\lambda \ln\left(\frac{L/2 + \sqrt{(L/2)^2+y^2}}{y}\right) \quad \text{(alternative form, } y > 0\text{)}.
\end{aligned}$$

4 $\vec{E}$ from potential: $E_y = -\frac{\mathrm{d}V}{\mathrm{d}y}$. Direct differentiation of the expression above yields exactly the field in 2. (Verification is straightforward.)

5 Comparison: The two methods give identical results, confirming consistency.

6 Limit $y \gg L$ for V: $\sqrt{(L/2)^2 + y^2} \approx y + \frac{L^2}{8y}$, and

$$V(y) \approx k\lambda \frac{L}{y} = \frac{kQ}{y},$$

the potential of a point charge Q.

7 Limit $y \gg L$ for E: $E(y) \approx \frac{kQ}{y^2}$, again the point–charge field.

8 This matches the field of a point charge $q = Q$ at distance y.

9 Change in potential energy for a proton moving from $y = 2L$ to $y = 6L$:

$$\begin{aligned}
L/2 &= 0.03\,\text{m}, \quad y_1 = 0.12\,\text{m}, \quad y_2 = 0.36\,\text{m}.\\
V(0.12) &= k\lambda \ln \frac{0.03 + \sqrt{0.03^2 + 0.12^2}}{-0.03 + \sqrt{0.03^2 + 0.12^2}} \approx 0.0360 \times 0.4949 = 0.017\,82\,\text{V},\\
V(0.36) &\approx 0.0360 \times 0.1664 = 0.005\,99\,\text{V}.\\
\Delta U &= e\big(V(0.12) - V(0.36)\big) = 1.602 \times 10^{-19} \times 0.01183 = 1.90 \times 10^{-21}\,\text{J}.
\end{aligned}$$

10 Proton speed at $y = 6L$ (starts from rest at $y = 2L$):

$$\frac{1}{2} m_p v^2 = \Delta U \implies v = \sqrt{\frac{2 \times 1.90 \times 10^{-21}}{1.6726 \times 10^{-27}}} \approx 1.51 \times 10^3\,\text{m/s}.$$

1.5 Exercise 11 – Electric Field due to a Uniformly Charged Disk

Given: $R = 2\,\text{cm} = 0.02\,\text{m}$, $\sigma = 6\,\mu\text{C/m}^2 = 6 \times 10^{-6}\,\text{C/m}^2$.

1 Charge element: $dq = \sigma\, dA$, where dA is an area element.

2 Total charge: $Q = \sigma \pi R^2 = 6 \times 10^{-6} \cdot \pi \cdot (0.02)^2 \approx 7.54 \times 10^{-9}\,\text{C}$.

3 Contribution of dA at point M on the z–axis:

$$d\vec{E} = \frac{k\, dq}{r^2} \hat{r}, \quad r = \sqrt{\rho^2 + z^2},$$

where ρ is the radial distance of dA from the center. The vertical component is

$$dE_z = \frac{k\, dq\, z}{(\rho^2 + z^2)^{3/2}}.$$

4 By symmetry, horizontal components cancel, leaving $\vec{E} = E_z\, \hat{k}$ (pointing away from the disk if $\sigma > 0$, i.e. $+z$ direction for $z > 0$).

5 Integration: $dq = \sigma \cdot 2\pi\rho \, d\rho$, so

$$E_z = \int_0^R \frac{k\,\sigma \cdot 2\pi\rho\, z}{(\rho^2 + z^2)^{3/2}}\, d\rho = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{z^2 + R^2}}\right) \quad (z > 0).$$

6 Potential (zero at infinity): $V(z) = \frac{\sigma}{2\varepsilon_0}\big(\sqrt{z^2 + R^2} - z\big)$.

7 Infinite plane limit ($R \gg z$): $E \approx \frac{\sigma}{2\varepsilon_0}$ (constant), $V \approx -\frac{\sigma}{2\varepsilon_0} z + \text{constant}$ (potential drops linearly).

8 **Plots:** $E(z)$ starts at $\sigma/(2\varepsilon_0)$ at $z = 0$ and decays to zero; $V(z)$ starts at $\sigma R/(2\varepsilon_0)$ at $z = 0$ and decreases approximately linearly for small z, then as $1/z$ for large z.

9 Far–field limit ($R \ll z$):

$$\sqrt{z^2 + R^2} \approx z + \frac{R^2}{2z} \implies E \approx \frac{\sigma}{2\varepsilon_0}\frac{R^2}{2z^2} = \frac{kQ}{z^2},$$

which is the field of a point charge Q at the origin.

2 Summary of Laws

2.1 Laws for Exercises 1, 3, and 6 (Discrete Charges)

- **Conservation of electric charge:** In an isolated system, the total net charge remains constant. Charge is quantized in multiples of $e = 1.602 \times 10^{-19}\,\mathrm{C}$.

- **Coulomb's law:** The force between two stationary point charges q_1 and q_2 separated by distance r is

  $$\vec{F}_{12} = k\,\frac{q_1 q_2}{r^2}\,\hat{r}_{12}, \quad k = \frac{1}{4\pi\varepsilon_0} \approx 8.99 \times 10^9\,\mathrm{N\,m^2/C^2}.$$

  Like charges repel, unlike charges attract.

- **Superposition principle:** The net force on a charge due to a collection of other charges is the vector sum of the individual Coulomb forces:

  $$\vec{F}_{\text{net}} = \sum_i \vec{F}_i.$$

- **Electric field:** Defined as $\vec{E} = \vec{F}/q_0$ (force per unit test charge). For a point charge,

  $$\vec{E} = k\frac{q}{r^2}\,\hat{r}.$$

  The field obeys superposition: $\vec{E}_{\text{net}} = \sum_i \vec{E}_i$.

- **Electric potential:** The potential at a point is the work per unit charge to bring a test charge from infinity to that point:

$$V(\vec{r}) = k\sum_i \frac{q_i}{|\vec{r}-\vec{r}_i|}, \quad V(\infty)=0.$$

- **Work and potential energy:** The work done by an external agent to move a charge q from A to B is

$$W_{\text{ext}} = q\big(V(B)-V(A)\big) = \Delta U.$$

  The electrostatic potential energy of a charge q at a point where the potential due to other charges is V is $U = qV$.

- **Energy of a system of discrete charges:**

$$U_{\text{system}} = \frac{1}{2}\sum_i q_i V_i = \sum_{i<j} k\frac{q_i q_j}{r_{ij}}.$$

- **Equilibrium of a charged particle:** Forces and torques are analyzed via free-body diagrams; for example, the equilibrium of a charged pendulum combines Coulomb repulsion, gravity, and tension.

- **Charging by rubbing, conduction, and induction** (qualitative understanding of charge transfer).

2.2 Laws for Exercises 10 and 11 (Continuous Charge Distributions)

- **Linear, surface, and volume charge densities:**

$$\lambda = \frac{dq}{dx}, \quad \sigma = \frac{dq}{dA}, \quad \rho = \frac{dq}{dV}.$$

- **Electric field of a continuous distribution:**

$$\vec{E}(\vec{r}) = k\int \frac{dq}{|\vec{r}-\vec{r}'|^3}\,(\vec{r}-\vec{r}') = k\int \frac{dq}{R^2}\,\hat{R},$$

  where $R = |\vec{r}-\vec{r}'|$ and the integral is over the charge distribution.

- **Potential of a continuous distribution ($V(\infty)=0$):**

$$V(\vec{r}) = k\int \frac{dq}{|\vec{r}-\vec{r}'|}.$$

- **Relationship between field and potential:** $\vec{E} = -\nabla V$. In one dimension, $E_y = -\frac{\mathrm{d}V}{\mathrm{d}y}$, etc.

- **Symmetry arguments:** Use symmetry to determine the direction of the field and simplify integrals (e.g., cancellation of horizontal components for a line or disk on the axis).

- **Standard integrals:**
  - $\displaystyle\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}}.$
  - $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\bigl(x+\sqrt{x^2+a^2}\bigr).$

- **Finite line charge:**

$$E_y(y) = \frac{k\lambda L}{y\sqrt{y^2+(L/2)^2}} = \frac{kQ}{y\sqrt{y^2+(L/2)^2}},$$
$$V(y) = k\lambda \ln\left(\frac{L/2+\sqrt{(L/2)^2+y^2}}{-L/2+\sqrt{(L/2)^2+y^2}}\right).$$

- **Uniformly charged disk:**

$$E_z(z) = \frac{\sigma}{2\varepsilon_0}\left(1-\frac{z}{\sqrt{z^2+R^2}}\right),$$
$$V(z) = \frac{\sigma}{2\varepsilon_0}\bigl(\sqrt{z^2+R^2}-z\bigr).$$

- **Limiting cases:**
  - Far away ($r \gg$ size): distribution reduces to a point charge: $E \approx kQ/r^2$, $V \approx kQ/r$.
  - Very close to a charged plane ($R \gg z$): field becomes uniform $E = \sigma/(2\varepsilon_0)$, potential linear in distance.

- **Work and energy in a continuous field:** $W = q\Delta V$, potential energy $U = qV$. Conservation of energy can be applied to find speeds (e.g., proton accelerated by a charged rod).