Solutions to Exercises 1, 2, 3

Classical Mechanics – Conductors in Static Equilibrium

Exercise 1 – Spherical Capacitor

Given data

- Inner solid sphere radius: $R = 1$ cm $= 0.01$ m
- Spherical shell inner radius: $R_1 = 3$ cm $= 0.03$ m
- Spherical shell outer radius: $R_2 = 5$ cm $= 0.05$ m
- Voltage between inner sphere and shell: $V = 24$ V (shell grounded)

1. Charge distribution

The inner sphere acquires a positive charge $+Q$. By induction, the inner surface of the shell (at R_1) gets charge $-Q$. Because the shell is grounded, its outer surface (at R_2) carries no net charge ($Q_{\text{outer}} = 0$).

2. Charge inside the inner sphere ($r < R$)

Inside a conductor in electrostatic equilibrium, $\mathbf{E} = 0$ and any net charge resides on the surface. Hence

$$\boxed{Q_{\text{inside}} = 0}.$$

3. Charge on the inner sphere surface ($r = R$)

Capacitance of a spherical capacitor (inner radius R, outer radius R_1):

$$C = 4\pi\varepsilon_0 \frac{RR_1}{R_1 - R}.$$

Numerically,

$$C = 4\pi(8.854 \times 10^{-12}) \frac{0.01 \times 0.03}{0.03 - 0.01} = 4\pi(8.854 \times 10^{-12}) \times 0.015 \approx 1.669 \times 10^{-12} \text{ F}.$$

Then

$$Q = CV = (1.669 \times 10^{-12}) \times 24 \approx 4.01 \times 10^{-11} \text{ C}.$$

$$\boxed{Q \approx 4.01 \times 10^{-11} \text{ C}}.$$

4. Charge on the inner surface of the shell ($r = R_1$)

By induction:

$$\boxed{-4.01 \times 10^{-11}\ \mathrm{C}}.$$

5. Charge on the outer surface of the shell ($r = R_2$)

Since the shell is grounded, the net charge on it must be such that the field outside vanishes. The inner surface already carries $-Q$; thus the outer surface carries zero charge:

$$\boxed{0}.$$

6. Surface charge densities

$$\begin{aligned}
\sigma(R) &= \frac{Q}{4\pi R^2} = \frac{4.01 \times 10^{-11}}{4\pi(0.01)^2} \approx 3.19 \times 10^{-8}\ \mathrm{C/m^2}, \\
\sigma(R_1) &= \frac{-Q}{4\pi R_1^2} \approx -3.54 \times 10^{-9}\ \mathrm{C/m^2}, \\
\sigma(R_2) &= 0.
\end{aligned}$$

Conclusion: The surface charge density is highest on the smallest radius and decreases as $1/r^2$.

7. Electric field $\mathbf{E}(r)$ from Gauss's law

- $r < R$: inside conductor, $E = 0$.
- $R < r < R_1$: Gaussian sphere encloses $+Q$.

$$E(r) \cdot 4\pi r^2 = \frac{Q}{\varepsilon_0} \Rightarrow E(r) = \frac{Q}{4\pi\varepsilon_0 r^2} \text{ (radially outward).}$$

- $R_1 < r < R_2$: inside conducting shell, $E = 0$.
- $r > R_2$: enclosed charge $Q + (-Q) + 0 = 0$, so $E = 0$.

Field lines start on the positive inner sphere and end on the negative inner surface of the shell.

Exercise 2 – Cylindrical Capacitor

Given data

- Inner cylinder radius: $r_1 = 1$ mm $= 0.001$ m
- Outer cylinder inner radius: $r_2 = 3$ mm $= 0.003$ m
- Outer cylinder outer radius: $r_3 = 4$ mm $= 0.004$ m
- Length: $L = 10$ cm $= 0.1$ m (treated as infinite)
- Voltage between cylinders: $V = 12$ V (outer conductor grounded)

1. Charge distribution

The inner cylinder carries $+Q$ (linear density $\lambda = Q/L$). The inner surface of the outer conductor (at $r = r_2$) gets induced charge $-Q$. Because the outer conductor is grounded and we require zero field outside, its outer surface (at $r = r_3$) carries zero charge.

2. Electric field from Gauss's law (cylindrical symmetry)

- $r < r_1$: inside conductor, $E = 0$.
- $r_1 < r < r_2$: Gaussian cylinder of radius r, length l encloses λl.

$$E(r) \cdot 2\pi r l = \frac{\lambda l}{\varepsilon_0} \;\Rightarrow\; E(r) = \frac{\lambda}{2\pi\varepsilon_0 r} \text{ (radially outward).}$$

- $r_2 < r < r_3$: inside conducting shell, $E = 0$.
- $r > r_3$: net enclosed charge $\lambda l + (-\lambda l) = 0$, so $E = 0$.

3. Potential difference between inner conductor and outer surface of B

$$V = \int_{r_1}^{r_2} E(r)\,dr = \frac{\lambda}{2\pi\varepsilon_0} \ln\frac{r_2}{r_1}.$$

Here $V = 12$ V is given, so λ can be determined if needed.

4. Plots of $E(r)$ and $V(r)$

- $E(r)$: zero for $r < r_1$; decays as $1/r$ in the gap; zero elsewhere.
- $V(r)$: constant inside inner cylinder (equal to 12 V); decreases logarithmically to zero at $r = r_2$; remains zero for $r \geq r_2$.

5. Capacitance of the coaxial capacitor

$$C = \frac{Q}{V} = \frac{\lambda L}{V} = \frac{2\pi\varepsilon_0 L}{\ln(r_2/r_1)}.$$

Numerically,

$$\ln\frac{r_2}{r_1} = \ln 3 \approx 1.0986, \quad C = \frac{2\pi(8.854 \times 10^{-12})(0.1)}{1.0986} \approx 5.06 \times 10^{-12} \text{ F} = 5.06 \text{ pF}.$$

6. Electrostatic energy stored

$$U = \frac{1}{2}CV^2 = \frac{1}{2}(5.06 \times 10^{-12})(144) \approx 3.64 \times 10^{-10} \text{ J}.$$

7. Approximation for small gap ($r_2 = r_1 + e,\ e \ll r_1$)

$$\ln\frac{r_2}{r_1} = \ln\left(1+\frac{e}{r_1}\right) \approx \frac{e}{r_1}, \quad \Rightarrow \quad C \approx \frac{2\pi\varepsilon_0 L r_1}{e} = \frac{\varepsilon_0(2\pi r_1 L)}{e}.$$

With $A = 2\pi r_1 L$ (the area of the inner cylinder), this becomes

$$C \approx \frac{\varepsilon_0 A}{e},$$

which is exactly the parallel-plate capacitor formula.

Exercise 3 – Equivalent Capacitance (Series-Parallel Combination)

Assumed configuration

C_1 and C_2 in series, their combination in parallel with C_3. Given: $C_1 = 3C,\ C_2 = 2C,\ C_3 = 4C$ with $C = 4\ \mu$F. Thus

$$C_1 = 12\ \mu\text{F}, \quad C_2 = 8\ \mu\text{F}, \quad C_3 = 16\ \mu\text{F}, \quad V_{AB} = 300\ \text{V}.$$

1. Equivalent capacitance

$$\begin{aligned} C_{12} &= \frac{C_1C_2}{C_1+C_2} = \frac{12\times 8}{12+8} = \frac{96}{20} = 4.8\ \mu\text{F},\\ C_{\text{eq}} &= C_{12} + C_3 = 4.8 + 16 = 20.8\ \mu\text{F}. \end{aligned}$$

2. Total charge stored

$$Q_{\text{total}} = C_{\text{eq}}V_{AB} = (20.8\times 10^{-6}) \times 300 = 6.24\times 10^{-3}\ \text{C} = 6240\ \mu\text{C}.$$

3. Charge on each capacitor

Parallel branch C_{12} and C_3 share the same voltage V_{AB}:

$$Q_3 = C_3V_{AB} = 16\times 300 = 4800\ \mu\text{C}.$$

The series combination C_{12} carries $Q_{12} = C_{12}V_{AB} = 4.8\times 300 = 1440\ \mu$C. In series, both capacitors have the same charge:

$$Q_1 = Q_2 = 1440\ \mu\text{C}.$$

4. Potential difference across each capacitor

$$V_1 = \frac{Q_1}{C_1} = \frac{1440}{12} = 120\ \text{V}, \qquad V_2 = \frac{Q_2}{C_2} = \frac{1440}{8} = 180\ \text{V}, \qquad V_3 = 300\ \text{V}.$$

(Check: $V_1 + V_2 = 300$ V as required.)

5. Energy stored in each capacitor

$$\begin{aligned} U_1 &= \tfrac{1}{2}C_1V_1^2 = 0.5 \times 12 \times 10^{-6} \times 14400 = 0.0864 \text{ J}, \\ U_2 &= 0.5 \times 8 \times 10^{-6} \times 32400 = 0.1296 \text{ J}, \\ U_3 &= 0.5 \times 16 \times 10^{-6} \times 90000 = 0.72 \text{ J}. \end{aligned}$$

Total energy: $U_{\text{total}} = 0.0864 + 0.1296 + 0.72 = 0.936$ J, which equals $\frac{1}{2}C_{\text{eq}}V_{AB}^2$.

Summary of Laws Used in the Exercises

Conductors in electrostatic equilibrium

- $\mathbf{E} = 0$ inside the conductor.
- Any net charge resides on the surface.
- The electric field just outside a conductor is perpendicular to the surface and has magnitude $E = \sigma/\varepsilon_0$.

Gauss's law

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{enc}}}{\varepsilon_0}.$$

Capacitance

$$C = \frac{Q}{V}.$$

- Spherical capacitor (inner radius a, outer radius b):
$$C = 4\pi\varepsilon_0 \frac{ab}{b-a}.$$
- Cylindrical capacitor (length L, inner radius a, outer radius b):
$$C = \frac{2\pi\varepsilon_0 L}{\ln(b/a)}.$$
- Parallel-plate capacitor (area A, separation d):
$$C = \frac{\varepsilon_0 A}{d}.$$
- With dielectric constant κ: $C = \kappa C_0$.

Combinations of capacitors

- Series: $\dfrac{1}{C_{\text{eq}}} = \sum \dfrac{1}{C_i}$, same charge on each.
- Parallel: $C_{\text{eq}} = \sum C_i$, same voltage across each.

Energy stored in a capacitor

$$U = \frac{1}{2}CV^2 = \frac{Q^2}{2C} = \frac{1}{2}QV.$$

Electric field from symmetry

- Spherical symmetry: $E(r) = \dfrac{Q_{\text{enc}}}{4\pi\varepsilon_0 r^2}$.
- Cylindrical symmetry: $E(r) = \dfrac{\lambda}{2\pi\varepsilon_0 r}$ (λ = linear charge density).

Potential difference

$$V_{ab} = \int_a^b \mathbf{E} \cdot d\mathbf{l}.$$