# Solutions and Summary of Laws – Magnetism (Problem Set N°04)

## 1 Solutions to Exercises

### 1.1 Exercise 1 – Muon in crossed $\vec{E}$ and $\vec{B}$ fields

1 **Magnetic field for no deflection.**
Lorentz force on a particle with charge $q = +e$: $\vec{F}_L = q(\vec{E} + \vec{v} \times \vec{B})$. For no deflection $\vec{F}_L = 0$ we need $\vec{E} = -\vec{v} \times \vec{B}$. Given $\vec{E}$ in $+x$, $\vec{v} = v\,\hat{j}$ $(+y)$, the cross product $\vec{v} \times \vec{B}$ must be in $-x$.

$$\vec{v}\times\vec{B} = v\,\hat{j}\times(B\hat{k}) = vB\,\hat{\imath}\ (+x) \implies \text{to get } -x,\ \vec{B} \text{ must be } -\hat{k} \text{ or } \vec{E} \text{ must be opposite.}$$

Assuming the figure shows the actual directions, the required magnitude is

$$E = vB \quad\implies\quad B = \frac{E}{v}.$$

$E = 10\,000$ V/cm $= 10^6$ V/m, $v = c/3 = 1.0 \times 10^8$ m/s.

$$B = \frac{10^6}{10^8} = 0.01\,\mathrm{T} = 10\,\mathrm{mT}.$$

The direction of $\vec{B}$ must be such that $\vec{v} \times \vec{B}$ opposes $\vec{E}$.

2 **Circular motion when $\vec{E}$ is turned off.**
Magnetic force provides the centripetal force:

$$qvB = \frac{mv^2}{R} \implies R = \frac{mv}{qB}.$$

Muon mass $m = 207\,m_e = 207 \times 9.109 \times 10^{-31}$ kg $\approx 1.886 \times 10^{-28}$ kg, $q = e = 1.602 \times 10^{-19}$ C, $B = 0.01$ T, $v = 1.0 \times 10^8$ m/s.

$$R \approx \frac{(1.886 \times 10^{-28})(1.0 \times 10^8)}{(1.602 \times 10^{-19})(0.01)} \approx 1.18 \times 10^{-1}\ \mathrm{m} = 11.8\,\mathrm{cm}.$$

The orbit is a clockwise circle in the plane perpendicular to $\vec{B}$ (if $\vec{B}$ along $+z$ and $\vec{v}$ along $+y$, the initial force is in $-x$, causing clockwise motion when viewed from $+z$).

## 1.2 Exercise 2 – Electron in a parallel-plate capacitor

1 **Direction of $\vec{E}$:** Electric field points from the positive plate (top) to the negative plate (bottom), i.e. vertically downward.

2 **Magnetic field for no deflection:** For an electron ($q = -e$) the Lorentz force is

$$\vec{F} = -e(\vec{E} + \vec{v} \times \vec{B}) = 0 \implies \vec{E} + \vec{v} \times \vec{B} = 0 \implies \vec{v} \times \vec{B} = -\vec{E}.$$

Let $\vec{v} = v\,\hat{\imath}$ (to the right), $\vec{E} = E\,(-\hat{\jmath})$ (downward). Then $-\vec{E} = E\,\hat{\jmath}$.

$$\vec{v} \times \vec{B} = v\,\hat{\imath} \times \vec{B}. \text{ For this to equal } E\,\hat{\jmath},\ \hat{\imath} \times \vec{B} \propto \hat{\jmath} \implies \vec{B} \parallel (-\hat{k}).$$

Indeed $\hat{\imath} \times (-\hat{k}) = \hat{\jmath}$. Therefore $\vec{B}$ must point into the page.

3 **Magnitude of $\vec{B}$:** $|\vec{v} \times \vec{B}| = vB = E$, so

$$B = \frac{E}{v} = \frac{\Delta V/d}{v} = \frac{\Delta V}{vd}.$$

## 1.3 Exercise 3 – Velocity selector and circular trajectory of $He^{+}$

1. **Velocity of the undeflected beam:**
   In crossed fields, $v = E/B$. With $E = 10^5$ V/m, $B = 0.04$ T,

$$v = \frac{10^5}{0.04} = 2.5 \times 10^6\,\text{m/s}.$$

2. **Trajectory in a uniform magnetic field:**
   The ions enter a region with only $\vec{B}$ perpendicular to $\vec{v}$. The magnetic force acts as centripetal force:

$$qvB = \frac{mv^2}{R} \implies R = \frac{mv}{qB}.$$

   $m_{\text{He}^+} = 6.6 \times 10^{-27}$ kg, $q = +e = 1.602 \times 10^{-19}$ C.

$$R = \frac{(6.6 \times 10^{-27})(2.5 \times 10^6)}{(1.602 \times 10^{-19})(0.04)} \approx 2.57\,\text{m}.$$

   The path is a circle (clockwise or counter-clockwise depending on the field direction).

## 1.4 Exercise 4 – Mass spectrometer

Doubly ionized atoms: charge $q = +2e$. Acceleration through a potential difference $V$:

$$\frac{1}{2}Mv^2 = qV = 2eV \implies v = \sqrt{\frac{4eV}{M}}.$$

In the uniform $\vec{B}$ field, perpendicular to $\vec{v}$:

$$qvB = \frac{Mv^2}{R} \implies M = \frac{qBR}{v} = \frac{2eBR}{\sqrt{4eV/M}}.$$

Squaring the relation $R = \frac{Mv}{qB}$ with $v = \sqrt{4eV/M}$:

$$R = \frac{M\sqrt{4eV/M}}{2eB} = \frac{\sqrt{4eVM}}{2eB} \implies R^2 = \frac{4eVM}{4e^2B^2} = \frac{MV}{eB^2}.$$

Thus

$$\boxed{M = \frac{eB^2R^2}{V}}.$$

## 1.5 Exercise 5 – Magnetic suspension of a wire

1. **Force per unit length between two parallel infinite wires:**

$$\frac{F}{L} = \frac{\mu_0 I_1 I_2}{2\pi d}.$$

If the currents are in the same direction the force is attractive; opposite directions give repulsion.

2. **Current in the suspended wire:**
The bottom wire (copper, diameter 4 mm) is held up by the magnetic attraction from the top wire ($I_1 = 100$ A, distance $d = 20$ cm). Weight per unit length of the bottom wire:

$$\frac{mg}{L} = \rho\pi\left(\frac{\phi}{2}\right)^2 g = 8.9\times10^3\times\pi\times(2\times10^{-3})^2\times9.8 \approx 1.096\,\mathrm{N/m}.$$

Equating magnetic force per unit length (upward) to weight:

$$\frac{\mu_0 I_1 I_2}{2\pi d} = \frac{mg}{L}.$$

$$I_2 = \frac{(mg/L)\,2\pi d}{\mu_0 I_1} = \frac{1.096\times2\pi\times0.2}{4\pi\times10^{-7}\times100} \approx 1.096\times10^4\ \mathrm{A} = 10.96\,\mathrm{kA}.$$

The current in the bottom wire must flow in the same direction as in the top wire to produce an attractive upward force.

## 1.6 Exercise 6 – Magnetic field of a circular loop

1. **Field on the axis of a single loop:**
By the Biot-Savart law,

$$\vec{B}(z) = \frac{\mu_0 I R^2}{2(R^2+z^2)^{3/2}}\,\hat{z},$$

directed along the loop's axis (perpendicular to the plane of the loop).

2. **Flat coil of $N$ turns:**

$$\vec{B}(z) = \frac{\mu_0 N I R^2}{2(R^2+z^2)^{3/2}}\,\hat{z}.$$

3. **Field at the centre ($z = 0$):**

$$B(0) = \frac{\mu_0 N I}{2R}.$$

4. **Ampere's law:** Ampere's law is not easily applied to a single circular loop because the magnetic field lacks the required symmetry; the result obtained from the Biot-Savart law is standard.

5. **Field lines:** The field resembles that of a magnetic dipole: lines emerge from one face, curve around, and enter the opposite face, forming closed loops.

## 1.7 Exercise 7 – Force between two current loops approximated as parallel wires

1 **Direction of $\vec{B}$ at the right-most point of loop 2:**
Loop 1 (top) carries clockwise current. At its right-most point the current points downward. For an infinite straight wire carrying current downward, the magnetic field at a point directly below the wire points to the right (east). Hence at the right-most point of loop 2 the field due to loop 1 is horizontal to the right.

2 **Magnitude of $\vec{B}$ (Ampere's law):**
Using the approximation of two parallel infinite wires separated by $d$, the field at the location of loop 2 is

$$B = \frac{\mu_0 I_0}{2\pi d}.$$

3 **Force on loop 2:**
Because the currents in the two loops are opposite at adjacent points (loop 1 clockwise, loop 2 counter-clockwise), the two loops behave as anti-parallel currents. The force per unit length between anti-parallel wires is repulsive and has magnitude $\mu_0 I_0^2/(2\pi d)$. Treating the entire loop as an effective length of wire equal to the circumference $2\pi R$, the total force is

$$F = \frac{\mu_0 I_0^2}{2\pi d} \times 2\pi R = \frac{\mu_0 I_0^2 R}{d} \quad \text{(repulsive)}.$$

The force pushes loop 2 downward (away from loop 1).

## 1.8 Exercise 8 – LHC dipole magnet

1 **Direction of $\vec{B}$ for clockwise protons:**
Protons moving clockwise in the ring's plane experience a centripetal force pointing toward the centre. For a positive charge, $\vec{F} = q\vec{v} \times \vec{B}$. If $\vec{v}$ is tangential clockwise,

$\vec{v} \times \vec{B}$ must point radially inward. The required magnetic field is vertical, pointing *out of the plane* of the ring (upward).

2 **Current in each loop for $B = 8$ T:**
The magnet is modelled as two rectangular loops (14 m×0.3 m) separated by $d = 10$ cm. The long edges (14 m) are effectively infinite straight wires. The two inner long edges carry opposite currents and produce the main field. Four long wires contribute: inner two at distance $d/2 = 0.05$ m, outer two at $d/2+0.3$ m $= 0.35$ m. The net field midway is

$$B = \frac{\mu_0 I}{\pi}\left(\frac{1}{d/2} - \frac{1}{d/2+w}\right), \quad w = 0.3 \text{ m}.$$

$$B = \frac{4\pi \times 10^{-7} I}{\pi}\left(\frac{1}{0.05} - \frac{1}{0.35}\right) = 4 \times 10^{-7} I \times (20 - 2.857) = 4 \times 10^{-7} I \times 17.143.$$

Setting $B = 8$ T:

$$I = \frac{8}{4 \times 10^{-7} \times 17.143} \approx 1.17 \times 10^6 \text{ A}.$$

This represents the total ampere-turns if the loops have many turns; with a single turn it is the current.

3 **Repulsion for anti-parallel currents:**
Consider two straight parallel wires carrying currents in opposite directions. The magnetic field produced by the first wire at the location of the second is perpendicular to the second wire. Using the right-hand rule, the Lorentz force $\vec{F} = I_2 \vec{L} \times \vec{B}$ points away from the first wire, showing repulsion.

4 **Energy stored in the field of all 1200 dipoles:**
Energy density: $u = \dfrac{B^2}{2\mu_0} = \dfrac{8^2}{2 \times 4\pi \times 10^{-7}} \approx 2.546 \times 10^7$ J/m$^3$. Volume of one dipole: cross-section 200 cm$^2$ = 0.02 m$^2$, length loop length 14 m; $V_{\text{one}} = 14 \times 0.02 = 0.28$ m$^3$. Total volume $V_{\text{tot}} = 1200 \times 0.28 = 336$ m$^3$. Total energy:

$$W = uV_{\text{tot}} \approx 2.546 \times 10^7 \times 336 \approx 8.56 \times 10^9 \text{ J}.$$

## 1.9 Exercise 9 – Magnetic field of a thin infinite current ribbon

The ribbon lies in the plane $z = 0$, carries a uniform current $I$ along the $+y$ direction, has width $W$ (along $x$ from 0 to $W$), and is infinitely long. We want the field at a point $P$ located in the plane of the ribbon, at a perpendicular distance $x$ from one edge (outside the ribbon, e.g. at $x = W + x_0$ with $x_0 = x$).

Each infinitesimal strip of width $dx'$ carries a current $dI = \frac{I}{W}dx'$ and acts as an infinite straight wire. The magnetic field of such a wire at a perpendicular distance $r$ is $\frac{\mu_0 dI}{2\pi r}$, its direction given by the right-hand rule. In the plane of the ribbon, for a current in $+y$, the field is perpendicular to the plane ($\pm z$).

For $P$ outside the ribbon on the right side (distance $x$ from the right edge):

$$B = \int_{x=0}^{W} \frac{\mu_0}{2\pi} \frac{I/W}{|x_{\text{edge}} + W - x'|}\, dx'\, (-\hat{k}),$$

where $x_{\text{edge}}$ is the distance from the left edge to the point... Let the right edge be at $x' = W$; then $P$ is at $x' = W + x$. Distances from the strips are $(W + x - x')$. All contributions add in the $-\hat{k}$ direction:

$$B = \frac{\mu_0 I}{2\pi W} \int_0^W \frac{dx'}{W + x - x'} = \frac{\mu_0 I}{2\pi W} \ln\frac{W + x}{x} \, (-\hat{k}).$$

If $P$ is on the left side (distance $x$ from the left edge), the field is in the $+\hat{k}$ direction:

$$B = \frac{\mu_0 I}{2\pi W} \ln\frac{W + x}{x} \, (+\hat{k}).$$

*Limit $W \to 0$:* Using $\ln(1 + W/x) \approx W/x$, we recover the field of a line current:

$$B \approx \frac{\mu_0 I}{2\pi W} \cdot \frac{W}{x} = \frac{\mu_0 I}{2\pi x},$$

which is the correct point-wire result.

## 2 Summary of Laws

### Lorentz force

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}).$$

For a charged particle moving in a uniform $\vec{B}$ with $\vec{v} \perp \vec{B}$, the motion is circular with radius

$$R = \frac{mv}{|q|B}, \qquad \omega = \frac{|q|B}{m}.$$

### Magnetic force on a current-carrying wire (Laplace force)

$$d\vec{F} = I \, d\vec{l} \times \vec{B}, \qquad \vec{F} = I \int d\vec{l} \times \vec{B}.$$

For a straight wire of length $L$ in a uniform $\vec{B}$: $F = ILB \sin\theta$.

### Biot–Savart law

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I \, d\vec{l} \times \hat{r}}{r^2}, \qquad \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{I \, d\vec{l'} \times (\vec{r} - \vec{r'})}{|\vec{r} - \vec{r'}|^3}.$$

### Ampere's law

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}}.$$

Useful for highly symmetric current distributions (infinite straight wire, solenoid, toroid).

## Magnetic field of an infinite straight wire

$$B = \frac{\mu_0 I}{2\pi r} \quad \text{(azimuthal)}.$$

## Force between two parallel infinite wires

$$\frac{F}{L} = \frac{\mu_0 I_1 I_2}{2\pi d} \begin{cases} \text{attractive if } I_1 \parallel I_2, \\ \text{repulsive if } I_1 \uparrow\downarrow I_2. \end{cases}$$

## Magnetic field on the axis of a circular loop (or flat coil)

$$B(z) = \frac{\mu_0 N I R^2}{2(R^2 + z^2)^{3/2}}, \qquad B_{\text{centre}} = \frac{\mu_0 N I}{2R}.$$

## Magnetic moment of a current loop

$$\vec{\mu} = N I A \,\hat{n}, \qquad \vec{\tau} = \vec{\mu} \times \vec{B}.$$

## Energy density of the magnetic field

$$u = \frac{B^2}{2\mu_0}.$$

## Superposition principle for $\vec{B}$

The total magnetic field is the vector sum of the fields produced by each individual current element.

## Key integrals

$$\int \frac{dx}{a - x} = -\ln|a - x|, \qquad \int \frac{dx}{\sqrt{x^2 + a^2}} = \ln\left(x + \sqrt{x^2 + a^2}\right),$$

$$\int \frac{dx}{(x^2 + a^2)^{3/2}} = \frac{x}{a^2\sqrt{x^2 + a^2}}.$$