General Electricity
Correction – Set 1

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DC Networks, Sinusoidal Steady State, Phasors & AC Power

General Electricity — Detailed Correction – Exercise Set 1

Focus on the Way of Thinking, Explicit Substitutions, and Step-by-Step Course Execution

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How to Use This Correction

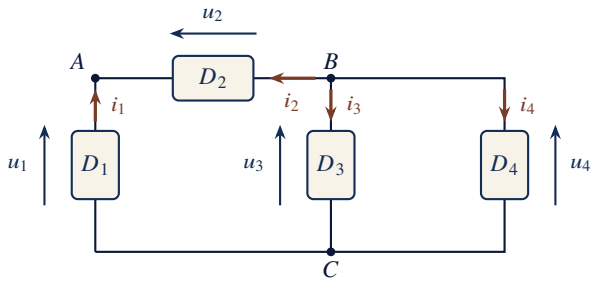
Each solution follows a strict four-stage analytical structure designed to train exam intuition: **Method** (identifies the exact course technique) ⟶ **Why we do this** (explains the mental train of thought and circuit clues) ⟶ **Execution** (complete line-by-line derivation with zero skipped steps and explicit **Sub** callouts) ⟶ **Pitfall / Sanity Check** (where students lose points and how to verify your answer).

Conventions & Phasor Notation: For $u_{AB} = V_A - V_B$, the voltage arrow points toward terminal A. In the sinusoidal steady state ($j^2 = -1$), $\bar{U} = Ue^{j\varphi_u} = U\angle\varphi_u$ and $\bar{I} = Ie^{j\varphi_i} = I\angle\varphi_i$ denote RMS complex phasors ($U = U_m/\sqrt{2}$, $I = I_m/\sqrt{2}$), and $\vec{U}(U, \varphi_u)$, $\vec{I}(I, \varphi_i)$ denote the associated RMS Fresnel vectors.

Difficulty: ★☆☆☆☆ Direct ★★☆☆☆ Standard ★★★☆☆ Exam level ★★★★☆ Tricky ★★★★★ Synthesis

Exercise 1

Configuration Summary: *Closed DC network of four dipoles D_1, D_2, D_3, D_4 between nodes A, B, C with $u_1 = V_A - V_C = 28\,\text{V}$, $u_2 = V_A - V_B = 12\,\text{V}$, $i_1 = 5\,\text{A}$ (directed $C \to A$), and $i_3 = 8\,\text{A}$ (directed $B \to C$).*



1) Determine the voltages u_3 and u_4 and the algebraic currents i_2 and i_4. ★☆☆☆☆

Method ***Kirchhoff's Voltage Law (KVL) on Independent Meshes and Kirchhoff's Current Law (KCL) at Nodes***

> ***Why we do this (Way of Thinking):***
> Because the internal constitutive laws of $D_1, \ldots, D_4$ are not specified, Ohm's law cannot be invoked. Instead, topological conservation laws determine every missing branch variable directly: traversing the left mesh $C \to A \to B \to C$ links u_3 to the known voltages u_1 and u_2, the parallel connection between B and C equates u_4 to u_3, KCL at the two-branch junction A fixes i_2 from i_1, and KCL at the three-branch node B isolates i_4.

Execution:

Step 1 — Voltages u_3 and u_4 via KVL: Orienting the left mesh clockwise ($C \to A \to B \to C$), arrow u_1 ($C \to A$) agrees with the traversal ($+u_1$), whereas arrow u_2 ($B \to A$) and arrow u_3 ($C \to B$) oppose the traversal ($-u_2$ and $-u_3$):

$$+u_1 - u_2 - u_3 = 0 \implies u_3 = u_1 - u_2 = 28 - 12 = 16\,\text{V}.$$

Orienting the right parallel mesh clockwise ($C \to B$ through D_3, then $B \to C$ through D_4):

$$+u_3 - u_4 = 0 \implies u_4 = u_3 = 16\,\text{V}.$$

Step 2 — Algebraic currents i_2 and i_4 via KCL: At junction A, both reference arrows i_1 (from C to A) and i_2 (from B to A) point into node A, while zero arrows leave node A:

$$\sum i_{\text{in}}(A) = \sum i_{\text{out}}(A) \implies i_1 + i_2 = 0 \implies i_2 = -i_1 = -5\,\text{A}.$$

At node B, all three reference arrows i_2 ($B \to A$), i_3 ($B \to C$), and i_4 ($B \to C$) leave node B:

$$\sum i_{\text{out}}(B) = \sum i_{\text{in}}(B) \implies i_2 + i_3 + i_4 = 0 \implies i_4 = -i_2 - i_3 = -(-5) - 8 = 5 - 8 = -3\,\text{A}.$$

$$u_3 = 16\,\text{V}, \qquad u_4 = 16\,\text{V}, \qquad i_2 = -5\,\text{A}, \qquad i_4 = -3\,\text{A}$$

> ► **Common Pitfall:**
> On the schematic, arrow u_2 points leftward ($u_2 = V_A - V_B$), not rightward. Writing $u_1 + u_2 - u_3 = 0$ by assuming u_2 drops from A to B yields $u_3 = 40\,\text{V}$ and violates $V_B - V_C = (V_A - V_C) - (V_A - V_B) = 28 - 12 = 16\,\text{V}$.

2) For each dipole D_k ($k = 1, 2, 3, 4$), identify whether its labeled arrows follow the receiver convention or the generator convention, compute $p_k = u_k\, i_k$, and state whether D_k absorbs or supplies electrical power. ★★☆☆☆

Method ***Receiver vs. Generator Sign Conventions and Algebraic Power Product $p_k = u_k\, i_k$***

> ***Why we do this (Way of Thinking):***
> The physical meaning of the algebraic product $p_k = u_k\, i_k$ depends entirely on the relative orientation of the reference arrows u_k and i_k across dipole D_k:
> - **Opposite arrow senses** (i_k enters at the arrowhead of u_k) $\implies$ **receiver convention**: $p_k > 0$ means D_k absorbs power; $p_k < 0$ means D_k supplies power.
> - **Identical arrow senses** (i_k exits from the arrowhead of u_k) $\implies$ **generator convention**: $p_k > 0$ means D_k supplies power; $p_k < 0$ means D_k absorbs power.

Execution: Evaluating each dipole from its branch arrows and the values found in question 1:

- **Dipole D_1:** u_1 points upward ($C \to A$) and i_1 points upward ($C \to A$, same sense) $\implies$ **generator convention**.
$$p_1 = u_1\, i_1 = (28\,\text{V})(5\,\text{A}) = +140\,\text{W} > 0 \implies D_1 \textbf{ supplies } 140\,\text{W}.$$
- **Dipole D_2:** u_2 points leftward ($B \to A$) and i_2 points leftward ($B \to A$, same sense) $\implies$ **generator convention**.
$$p_2 = u_2\, i_2 = (12\,\text{V})(-5\,\text{A}) = -60\,\text{W} < 0 \implies D_2 \textbf{ absorbs } 60\,\text{W}.$$
- **Dipole D_3:** u_3 points upward ($C \to B$) whereas i_3 points downward ($B \to C$, opposite senses) $\implies$ **receiver convention**.
$$p_3 = u_3\, i_3 = (16\,\text{V})(8\,\text{A}) = +128\,\text{W} > 0 \implies D_3 \textbf{ absorbs } 128\,\text{W}.$$
- **Dipole D_4:** u_4 points upward ($C \to B$) whereas i_4 points downward ($B \to C$, opposite senses) $\implies$ **receiver convention**.
$$p_4 = u_4\, i_4 = (16\,\text{V})(-3\,\text{A}) = -48\,\text{W} < 0 \implies D_4 \textbf{ supplies } 48\,\text{W}.$$

Dipole	u_k (V)	i_k (A)	Arrow Convention	$p_k = u_k\, i_k$ (W)	Physical Behavior
D_1	+28	+5	Generator (same sense)	+140	Supplies 140 W
D_2	+12	−5	Generator (same sense)	−60	Absorbs 60 W
D_3	+16	+8	Receiver (opposite sense)	+128	Absorbs 128 W
D_4	+16	−3	Receiver (opposite sense)	−48	Supplies 48 W

$$D_1:\ \text{supplies } 140\,\text{W}, \quad D_2:\ \text{absorbs } 60\,\text{W}, \quad D_3:\ \text{absorbs } 128\,\text{W}, \quad D_4:\ \text{supplies } 48\,\text{W}$$

> ► **Common Pitfall:**
> A dipole labeled in the generator convention is not necessarily an active power supplier: because $i_2 = -5\,\text{A} < 0$, actual charge flows from A to B (entering the higher-potential terminal A of D_2), so D_2 behaves physically as a receiver absorbing 60 W (such as a battery undergoing recharge).

3) Dipole D_4 is re-labeled in the generator convention while keeping $u_4 = V_B - V_C$ unchanged. Give the new reference current i_4', compute $p_{4,\text{gen}} = u_4\, i_4'$, and compare the physical conclusion with question 2.

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Method *Convention Reversal Invariance ($i_4' = -i_4$, $p_{4,\text{gen}} = -p_{4,\text{rec}}$)*

> ***Why we do this (Way of Thinking):***
> Holding the voltage arrow $u_4 = V_B - V_C$ upward ($C \to B$), switching D_4 from the receiver convention to the generator convention requires reversing its current reference arrow so that i_4' also points upward ($C \to B$). Reversing a reference arrow negates the algebraic value of the current while leaving the physical charge flow unaltered.

Execution: Since i_4' points from C to B opposite to i_4 ($B \to C$):

$$i_4' = -i_4 = -(-3\,\text{A}) = +3\,\text{A}.$$

The power product in the generator convention is:

$$p_{4,\text{gen}} = u_4\, i_4' = (16\,\text{V})(+3\,\text{A}) = +48\,\text{W} > 0.$$

In the generator convention, $p_{4,\text{gen}} > 0$ signifies that D_4 **supplies** 48 W, matching the physical conclusion of question 2.

$$i_4' = +3\,\text{A}, \qquad p_{4,\text{gen}} = +48\,\text{W} \text{ (supplies 48 W, invariant under re-labeling)}$$

> ✓ **Sanity Check:**
> For any two-terminal element, $p_{\text{gen}} = -p_{\text{rec}}$ identically: here $p_{4,\text{gen}} = +48\,\text{W} = -(-48\,\text{W}) = -p_{4,\text{rec}}$. Physical energy transfer depends only on actual potentials and actual charge motion, never on human arrow labels.

4) Verify power conservation across the complete network. ★☆☆☆☆

Method *Global Power Balance (Tellegen's Theorem)*

> ***Why we do this (Way of Thinking):***
> In any isolated lumped electrical network obeying KCL and KVL, electromagnetic energy cannot accumulate at ideal wires or nodes, so the sum of electrical powers supplied by active elements equals the sum of electrical powers absorbed by passive/receiving elements at every instant.

Execution: Summing the supplied powers (D_1 and D_4) and the absorbed powers (D_2 and D_3) separately from question 2:

$$\sum P_{\text{supplied}} = P_{\text{sup}}(D_1) + P_{\text{sup}}(D_4) = 140 + 48 = 188\,\text{W},$$
$$\sum P_{\text{absorbed}} = P_{\text{abs}}(D_2) + P_{\text{abs}}(D_3) = 60 + 128 = 188\,\text{W}.$$

Equivalently, converting every dipole uniformly to the receiver convention ($p_{1,\text{rec}} = -p_1 = -140\,\text{W}$, $p_{2,\text{rec}} = -p_2 = +60\,\text{W}$, $p_{3,\text{rec}} = +128\,\text{W}$, $p_{4,\text{rec}} = -48\,\text{W}$):

$$\sum_{k=1}^{4} p_{k,\text{rec}} = (-140) + 60 + 128 + (-48) = 0\,\text{W}.$$

$$\sum P_{\text{supplied}} = \sum P_{\text{absorbed}} = 188\,\text{W} \iff \sum_{k=1}^{4} p_{k,\text{rec}} = 0\,\text{W}$$

✓ **Sanity Check:**
Checking $\sum_{k=1}^{4} p_{k,\text{rec}} = 0$ is the fastest exam diagnostic for KCL/KVL arithmetic: even a single sign error in u_3, i_2, or i_4 breaks the $188\,\text{W} = 188\,\text{W}$ balance immediately.

Exercise 2

Configuration Summary: *Active DC network across (A, B) formed by Branch I $[(e_1 = 30\,\text{V}, r_1 = 2.5\,\Omega)$ in series opposition with $(e_2 = 6\,\text{V}, r_2 = 1.5\,\Omega)]$ in parallel with Branch II $[(\eta_B = 2\,\text{A}, r_B = 12\,\Omega)]$, feeding load $R_L = 5\,\Omega$.*

1) Reduce Branch I to a single real voltage generator (e_A, r_A), then convert it into its equivalent current generator (η_A, r_A). ★☆☆☆☆

Method ***Series Association of Voltage Generators and Thévenin-to-Norton Conversion***

Why we do this (Way of Thinking):
Voltage generators connected in series combine directly by algebraic addition of their electromotive forces (taking signs along the chosen branch polarity $B \to A$) and arithmetic addition of their internal resistances. Once Branch I is reduced to a single real voltage source (e_A, r_A), converting it to its Norton current form $(\eta_A = e_A/r_A, r_A)$ prepares it to be merged in parallel with Branch II.

Execution:

Step 1 — Single real voltage generator (e_A, r_A) of Branch I: Tracing Branch I from node B to node A, arrow e_1 points toward A $(+e_1)$ while arrow e_2 points toward B $(-e_2)$. Internal resistances always add positively regardless of electromotive force polarity:

$$e_A = e_1 - e_2 = 30 - 6 = 24\,\text{V} \quad \text{(arrow directed from } B \text{ toward } A\text{)},$$
$$r_A = r_1 + r_2 = 2.5 + 1.5 = 4.0\,\Omega.$$

Step 2 — Equivalent current generator (η_A, r_A) of Branch I:

Sub **Apply the Thévenin–Norton transformation $\eta_A = e_A/r_A$ with unchanged parallel internal resistance r_A:**

$$\eta_A = \frac{e_A}{r_A} = \frac{24\,\text{V}}{4.0\,\Omega} = 6.0\,\text{A} \quad \text{(directed from } B \text{ toward } A, \text{ in parallel with } r_A = 4.0\,\Omega\text{)}.$$

$$(e_A,\, r_A) = (24\,\text{V},\, 4.0\,\Omega), \qquad (\eta_A,\, r_A) = (6.0\,\text{A},\, 4.0\,\Omega)$$

► **Common Pitfall:**
Never subtract internal resistances when two voltage generators oppose one another ($r_A \neq r_1 - r_2 = 1.0\,\Omega$). Both passive internal resistors dissipate Joule heat in series and therefore add unconditionally ($r_A = r_1 + r_2 = 4.0\,\Omega$).

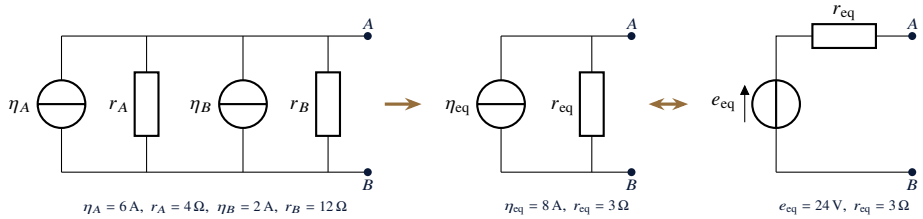
2) Deduce the single equivalent current generator $(\eta_{\text{eq}}, r_{\text{eq}})$ and the single equivalent voltage generator $(e_{\text{eq}}, r_{\text{eq}})$ of the entire active network left of terminals (A, B). Give the open-circuit voltage U_{oc} and the

short-circuit current I_{sc} across (A, B). ★★☆☆☆

Method ***Parallel Association of Norton Generators and Norton-to-Thévenin Conversion***

> ***Why we do this (Way of Thinking):***
> Dual to series voltage sources, parallel current generators combine by algebraic addition of their short-circuit currents ($\eta_{eq} = \eta_A + \eta_B$) and addition of their parallel internal conductances ($G_{eq} = 1/r_A + 1/r_B$). Converting the resulting Norton pair back to $(e_{eq} = r_{eq}\eta_{eq}, r_{eq})$ immediately exposes the open-circuit voltage $U_{oc} = e_{eq}$ and short-circuit current $I_{sc} = \eta_{eq}$.

Execution:



Step 1 — Single equivalent current generator (η_{eq}, r_{eq}): Both $\eta_A = 6.0\,\text{A}$ and $\eta_B = 2.0\,\text{A}$ pump current from rail B toward rail A:

$$\eta_{eq} = \eta_A + \eta_B = 6.0 + 2.0 = 8.0\,\text{A}.$$

Combining the parallel internal resistances $r_A = 4.0\,\Omega$ and $r_B = 12\,\Omega$:

$$\frac{1}{r_{eq}} = \frac{1}{r_A} + \frac{1}{r_B} = \frac{1}{4.0} + \frac{1}{12} = \frac{3+1}{12} = \frac{4}{12} = \frac{1}{3}\,\text{S} \implies r_{eq} = \frac{r_A r_B}{r_A + r_B} = \frac{4.0 \times 12}{4.0 + 12} = \frac{48}{16} = 3.0\,\Omega.$$

Step 2 — Single equivalent voltage generator (e_{eq}, r_{eq}), U_{oc}, and I_{sc}:

$$e_{eq} = r_{eq}\,\eta_{eq} = (3.0\,\Omega)(8.0\,\text{A}) = 24\,\text{V}, \qquad r_{eq} = 3.0\,\Omega.$$

At open circuit ($I_L = 0$), zero voltage drops across r_{eq}, so $U_{oc} = e_{eq} = 24\,\text{V}$. Under a terminal short circuit ($U_{AB} = 0$), zero current diverts into the parallel shunt r_{eq}, so $I_{sc} = \eta_{eq} = 8.0\,\text{A}$.

$$(\eta_{eq}, r_{eq}) = (8.0\,\text{A}, 3.0\,\Omega), \quad (e_{eq}, r_{eq}) = (24\,\text{V}, 3.0\,\Omega), \quad U_{oc} = 24\,\text{V}, \quad I_{sc} = 8.0\,\text{A}$$

> ✓ **Sanity Check:**
> Notice why $e_{eq} = 24\,\text{V}$ equals both $e_A = 24\,\text{V}$ and $r_B\eta_B = 12 \times 2 = 24\,\text{V}$: Branch I and Branch II happen to have identical open-circuit voltages (24 V), so placing them in parallel leaves the open-circuit voltage unchanged at 24 V while lowering the internal resistance from $4.0\,\Omega$ and $12\,\Omega$ down to $4 \parallel 12 = 3.0\,\Omega$.

3) Write the external characteristic $U_{AB}(I_L)$ and determine the operating point (I_L, U_L) and load power P_L when $R_L = 5\,\Omega$ is connected across (A, B), checking that both equivalent models yield identical results.
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Method ***Pouillet's Single-Loop Law vs. Two-Branch Current Divider***

> ***Why we do this (Way of Thinking):***
> Connecting $R_L = 5\,\Omega$ imposes the load line $U_L = R_L I_L$ simultaneously with the active dipole's linear terminal law $U_{AB}(I_L) = e_{eq} - r_{eq} I_L$. Evaluating (I_L, U_L, P_L) first with the series voltage model and then with the parallel current-divider model confirms external terminal equivalence.

Execution:

Step 1 — External characteristic and solution via the Thévenin voltage model: In the generator convention for the active network across (A, B):

$$U_{AB}(I_L) = e_{\text{eq}} - r_{\text{eq}}\, I_L = 24 - 3.0\, I_L \quad (U_{AB} \text{ in V},\ I_L \text{ in A}).$$

Intersecting with the passive load characteristic $U_L = U_{AB} = R_L\, I_L = 5.0\, I_L$:

$$24 - 3.0\, I_L = 5.0\, I_L \implies 8.0\, I_L = 24 \implies I_L = \frac{24}{8.0} = 3.0\,\text{A},$$
$$U_L = R_L\, I_L = (5.0\,\Omega)(3.0\,\text{A}) = 15\,\text{V},$$
$$P_L = U_L\, I_L = (15\,\text{V})(3.0\,\text{A}) = 45\,\text{W}.$$

Step 2 — Cross-check via the Norton current-generator model: With $\eta_{\text{eq}} = 8.0\,\text{A}$ feeding $r_{\text{eq}} = 3.0\,\Omega$ in parallel with $R_L = 5.0\,\Omega$, the two-branch current divider gives:

$$I_L = \eta_{\text{eq}}\,\frac{r_{\text{eq}}}{r_{\text{eq}} + R_L} = 8.0 \times \frac{3.0}{3.0 + 5.0} = 8.0 \times \frac{3.0}{8.0} = 3.0\,\text{A},$$

whence $U_L = R_L I_L = 5.0 \times 3.0 = 15\,\text{V}$ and $P_L = R_L I_L^2 = 5.0 \times 9.0 = 45\,\text{W}$, identical in every respect.

$$U_{AB}(I_L) = 24 - 3\, I_L, \qquad I_L = 3.0\,\text{A}, \qquad U_L = 15\,\text{V}, \qquad P_L = 45\,\text{W}$$

> ✓ **Sanity Check:**
> In the two-branch resistance form of the current divider, the numerator is always the *opposite* parallel resistance $r_{\text{eq}} = 3.0\,\Omega$ (not $R_L = 5.0\,\Omega$), because a smaller shunt $r_{\text{eq}} < R_L$ diverts the larger share of η_{eq} ($I_{r_{\text{eq}}} = 5.0\,\text{A} > I_L = 3.0\,\text{A}$).

4) With (A, B) left in open circuit ($I_L = 0$), compute the internal Joule power dissipated inside the single equivalent voltage model $(e_{\text{eq}}, r_{\text{eq}})$ and inside the single equivalent current model $(\eta_{\text{eq}}, r_{\text{eq}})$. Comment.

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Method ***Internal Joule Loss Comparison Under Open-Circuit Conditions***

> ***Why we do this (Way of Thinking):***
> Two dipoles are electrically equivalent if and only if they enforce the same terminal relationship $U_{AB}(I_L)$ as viewed from outside (A, B). Computing $r_{\text{eq}} i_{\text{int}}^2$ inside each model at $I_L = 0$ tests whether terminal equivalence extends to internal energy dissipation.

Execution: When terminals (A, B) are left open ($I_L = 0\,\text{A}$):

- **Inside the equivalent voltage model $(e_{\text{eq}}, r_{\text{eq}})$:** The series current through r_{eq} equals $I_L = 0\,\text{A}$, so

$$P_{\text{int,Th}} = r_{\text{eq}}\, I_L^2 = (3.0\,\Omega)(0)^2 = 0\,\text{W}.$$

- **Inside the equivalent current model $(\eta_{\text{eq}}, r_{\text{eq}})$:** Because no current exits at (A, B), the full source current $\eta_{\text{eq}} = 8.0\,\text{A}$ circulates internally through the parallel resistor $r_{\text{eq}} = 3.0\,\Omega$, dissipating

$$P_{\text{int,N}} = r_{\text{eq}}\, \eta_{\text{eq}}^2 = (3.0\,\Omega)(8.0\,\text{A})^2 = 3.0 \times 64 = 192\,\text{W}.$$

Comment: Thévenin–Norton equivalence holds **strictly externally** at terminals (A, B) (for any load connected outside (A, B)) and **never** applies to internal currents or internal Joule losses inside the replaced active box. (Indeed, in the actual two-branch physical network at $I_L = 0$, $U_{AB} = 24\,\text{V} = e_A$ drives zero current in Branch I while $r_B = 12\,\Omega$ carries $\eta_B = 2.0\,\text{A}$, dissipating $r_B \eta_B^2 = 48\,\text{W}$ — different from both 0 W and 192 W.)

$$P_{\text{int,Th}}(I_L{=}0) = 0\,\text{W} \neq P_{\text{int,N}}(I_L{=}0) = 192\,\text{W} \text{ (equivalence is strictly external)}$$

> ► **Common Pitfall:**
> Never compute the power dissipated in an internal resistor of the original circuit by using the reduced Thévenin or Norton model; once a sub-network has been collapsed into $(e_{\text{eq}}, r_{\text{eq}})$, internal branch powers can only be evaluated by returning to the original circuit topology.

5) Explain why an ideal voltage source ($r = 0\,\Omega$) cannot be transformed into an equivalent current source. ★☆☆☆☆

Method ***Singular Limit $r \to 0^+$ of the Linear Terminal Characteristic***

> ***Why we do this (Way of Thinking):***
> The Thévenin–Norton bijection relies on solving $U_{AB} = e - r\,I_L$ for $I_L = \frac{e}{r} - \frac{1}{r}U_{AB}$, which requires dividing by $r \neq 0$. Inspecting both the algebraic formula and the geometric (I_L, U_{AB}) characteristic as $r \to 0^+$ reveals why the inversion breaks down.

Execution: An ideal voltage generator ($r = 0\,\Omega$, $e \neq 0$) enforces the fixed terminal voltage

$$U_{AB} = e \qquad \text{for every finite load current } I_L \in (-\infty, +\infty).$$

Attempting to apply $\eta = e/r$ and $G = 1/r$ in the limit $r \to 0^+$ causes both the short-circuit current $\eta = e/r \to \infty$ and the internal parallel conductance $G = 1/r \to \infty$ to diverge, turning the current equation $I_L = \eta - G\,U_{AB}$ into the indeterminate expression $\infty - \infty$. Because U_{AB} is locked to e rather than varying linearly with I_L, no finite current generator (η, r) can reproduce an ideal voltage source.

$$r = 0\,\Omega \implies U_{AB} = e\ (\forall I_L),\ \eta = \frac{e}{r} \to \infty \text{ (no finite Norton equivalent exists)}$$

> ✓ **Sanity Check:**
> By exact duality, an ideal current source ($r \to \infty$, $G = 0$, fixing $I_L = \eta$ for arbitrary U_{AB}) cannot be transformed into an equivalent voltage source because $e = r\,\eta \to \infty$.

Exercise 3

Configuration Summary: *DC steady-state network with $R_s = 4\,\Omega$, $L = 50\,\text{mH}$ (between D and A), three parallel branches across (A, B) [$R_1 = 6\,\Omega$, $R_2 = 12\,\Omega$ carrying $I_2 = 2.0\,\text{A}$, and $R_{3a} = 1\,\Omega$ in series with $R_{3b} = 3\,\Omega$ tapped at M], and an upper bridge ($R_c = 7\,\Omega$, $C = 20\,\mu\text{F}$) connecting D to M.*

1) State the behavior of the ideal coil L and ideal capacitor C in the DC steady state, and give the values of u_L and i_C. ★☆☆☆☆

Method ***Time-Invariant Differential Laws ($d/dt = 0$) of Ideal Reactive Dipoles***

> ***Why we do this (Way of Thinking):***
> In the DC steady state (long after any switching transient has decayed), every current and voltage in the network is constant in time, so all time derivatives vanish identically ($di_L/dt = 0$ and $du_C/dt = 0$). Substituting $d/dt = 0$ into the constitutive laws of L and C reduces the reactive network to a purely resistive DC circuit.

Execution:

- **Ideal coil $L = 50\,\text{mH}$:** With $i_L = \text{const.}$, its terminal voltage in the receiver convention is

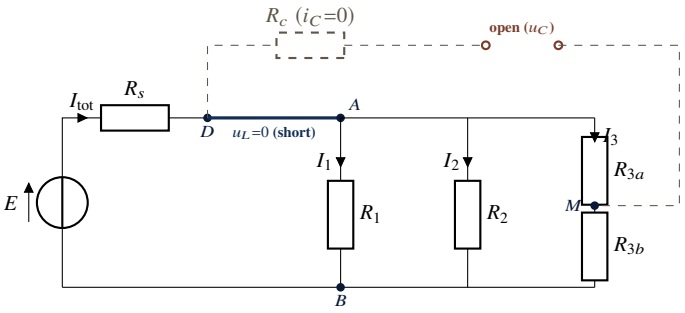
$$u_L = L\,\frac{di_L}{dt} = (50 \times 10^{-3}\,\text{H})(0) = 0\,\text{V},$$

so the ideal coil behaves as an **ideal short circuit** (closed wire connecting node D to node A, giving $V_D = V_A$).

- **Ideal capacitor $C = 20\,\mu\text{F}$:** With $u_C = \text{const.}$, the current through its branch is

$$i_C = C\,\frac{du_C}{dt} = (20\times 10^{-6}\,\text{F})(0) = 0\,\text{A},$$

so the ideal capacitor behaves as an **ideal open circuit** (interrupting all DC current along the upper bridge D–R_c–C–M).



$$L \longrightarrow \text{short circuit } (u_L = 0\,\text{V},\ V_D = V_A), \qquad C \longrightarrow \text{open circuit } (i_C = 0\,\text{A})$$

> ► **Common Pitfall:**
> An open-circuited capacitor in DC has $i_C = 0\,\text{A}$, **not** $u_C = 0\,\text{V}$. Its plates remain electrostatically polarized at whatever steady potential difference $V_D - V_M$ the surrounding resistive network imposes across them.

2) Calculate the equivalent conductance G_p and equivalent resistance R_p of the three-branch parallel block between nodes A and B. ★☆☆☆☆

Method *Unloaded Series Branch Reduction Followed by Parallel Conductance Summation*

> ***Why we do this (Way of Thinking):***
> Before combining the three branches between A and B, one must check whether tap node M diverts any current out of the third branch. Because $i_C = 0\,\text{A}$, KCL at node M guarantees that R_{3a} and R_{3b} carry the exact same current I_3, making them strictly in series ($R_3 = R_{3a} + R_{3b}$). Summing the three branch conductances $G_k = 1/R_k$ avoids nested reciprocals.

Execution: Since $i_C = 0\,\text{A}$ at tap M, the third parallel branch has total series resistance:

$$R_3 = R_{3a} + R_{3b} = 1 + 3 = 4.0\,\Omega.$$

Adding the conductances of the three parallel branches ($R_1 = 6\,\Omega$, $R_2 = 12\,\Omega$, $R_3 = 4\,\Omega$) across (A, B):

$$G_p = G_1 + G_2 + G_3 = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} = \frac{1}{6} + \frac{1}{12} + \frac{1}{4} = \frac{2+1+3}{12} = \frac{6}{12} = 0.50\,\text{S}.$$

Inverting G_p yields the equivalent parallel resistance:

$$R_p = \frac{1}{G_p} = \frac{1}{0.50\,\text{S}} = 2.0\,\Omega.$$

$$G_p = 0.50\,\text{S}, \qquad R_p = 2.0\,\Omega$$

> ✓ **Sanity Check:**
> The equivalent resistance of parallel positive resistors is strictly smaller than the smallest individual branch resistance: $R_p = 2.0\,\Omega < \min(6,\ 12,\ 4)\,\Omega = 4.0\,\Omega$.

3) From $I_2 = 2.0\,\text{A}$, determine the voltage V_{AB}, the total current I_{tot}, the branch currents I_1 and I_3, and the source electromotive force E. Verify I_1, I_2, and I_3 using the current divider rule. ★★☆☆☆

Method *Ohm's Law Back-Propagation and n-Branch Conductance Current Divider*

> ***Why we do this (Way of Thinking):***
> Knowing the current $I_2 = 2.0\,\text{A}$ through a single known resistor $R_2 = 12\,\Omega$ immediately pins down the common voltage $V_{AB} = R_2 I_2$ across all three parallel branches. From V_{AB}, every parallel branch current (I_1, I_3), the total line current $I_{\text{tot}} = V_{AB}/R_p$, and finally the upstream source electromotive force $E = R_s I_{\text{tot}} + V_{AB}$ follow without solving any simultaneous linear system.

Execution:

Step 1 — Parallel block voltage V_{AB}, currents $(I_{\text{tot}}, I_1, I_3)$, and source E:

$$V_{AB} = R_2\, I_2 = (12\,\Omega)(2.0\,\text{A}) = 24\,\text{V},$$
$$I_1 = \frac{V_{AB}}{R_1} = \frac{24\,\text{V}}{6\,\Omega} = 4.0\,\text{A}, \qquad I_3 = \frac{V_{AB}}{R_3} = \frac{24\,\text{V}}{4.0\,\Omega} = 6.0\,\text{A}.$$

Since $i_C = 0\,\text{A}$ leaves node D, the entire source current I_{tot} enters node A:

$$I_{\text{tot}} = \frac{V_{AB}}{R_p} = \frac{24\,\text{V}}{2.0\,\Omega} = 12.0\,\text{A} \quad (\text{matching } I_1 + I_2 + I_3 = 4.0 + 2.0 + 6.0 = 12.0\,\text{A}).$$

Applying KVL around the outer loop ($B \to E \to R_s \to D \to L \to A \to B$, with $u_L = 0\,\text{V}$):

$$E - R_s\, I_{\text{tot}} - u_L - V_{AB} = 0 \implies E = R_s\, I_{\text{tot}} + V_{AB} = (4\,\Omega)(12.0\,\text{A}) + 24\,\text{V} = 48 + 24 = 72\,\text{V}.$$

Step 2 — Verification via the general three-branch current divider:

Sub **For n parallel branches of total conductance $G_p = 1/R_p$, branch k carries $I_k = I_{\text{tot}}\,(G_k/G_p) = I_{\text{tot}}\,(R_p/R_k)$:**

$$I_1 = I_{\text{tot}}\,\frac{R_p}{R_1} = 12.0 \times \frac{2.0}{6.0} = 4.0\,\text{A},$$
$$I_2 = I_{\text{tot}}\,\frac{R_p}{R_2} = 12.0 \times \frac{2.0}{12.0} = 2.0\,\text{A},$$
$$I_3 = I_{\text{tot}}\,\frac{R_p}{R_3} = 12.0 \times \frac{2.0}{4.0} = 6.0\,\text{A}.$$

$$V_{AB} = 24\,\text{V}, \qquad I_{\text{tot}} = 12.0\,\text{A}, \qquad I_1 = 4.0\,\text{A}, \qquad I_3 = 6.0\,\text{A}, \qquad E = 72\,\text{V}$$

> ▸ **Common Pitfall:**
> Never apply the two-resistor formula $I_1 = I_{\text{tot}}\frac{R_2}{R_1+R_2}$ directly to three parallel branches; for $n \geq 3$ branches, one must always use conductances $I_k = I_{\text{tot}}\frac{G_k}{G_1+G_2+G_3}$ (or equivalently $I_k = I_{\text{tot}}\frac{R_p}{R_k}$).

4) Determine the potential $V_M - V_B$, the steady-state capacitor voltage $u_C = V_D - V_M$ (across the plates of C), and the electric charge q_C stored on C. ★★☆☆☆

Method *Unloaded Voltage Divider at Tap M and KVL Across the Open Capacitor Bridge*

> ***Why we do this (Way of Thinking):***
> Because $i_C = 0\,\text{A}$ enters tap M, the series pair (R_{3a}, R_{3b}) forms an unloaded voltage divider across V_{AB}, giving $V_M - V_B$ immediately. Likewise, $i_C = 0\,\text{A}$ causes zero ohmic drop across the series resistor R_c ($u_{R_c} = R_c i_C = 0$), while $u_L = 0$ equates V_D to V_A, so the entire potential difference $V_D - V_M$ appears directly across the plates of C.

Execution:

Step 1 — Tap potential $V_M - V_B$ via the unloaded voltage divider:

$$V_M - V_B = V_{AB}\,\frac{R_{3b}}{R_{3a}+R_{3b}} = 24\,\text{V} \times \frac{3}{1+3} = 24 \times \frac{3}{4} = 18\,\text{V} \quad (\text{also } R_{3b}\,I_3 = 3 \times 6.0 = 18\,\text{V}).$$

Step 2 — Steady-state capacitor voltage u_C and stored charge q_C: Since $u_L = V_D - V_A = 0\,\text{V}$, node D sits at $V_D - V_B = V_A - V_B = V_{AB} = 24\,\text{V}$. Along the upper bridge from D through R_c and C to M, $u_{R_c} = R_c\,i_C = (7\,\Omega)(0\,\text{A}) = 0\,\text{V}$, so KVL yields:

$$V_D - V_M = u_{R_c} + u_C = 0 + u_C \implies u_C = (V_D - V_B) - (V_M - V_B) = 24 - 18 = 6.0\,\text{V}.$$

Multiplying by $C = 20\,\mu\text{F} = 20 \times 10^{-6}\,\text{F}$ gives the stored charge:

$$q_C = C\,u_C = (20 \times 10^{-6}\,\text{F})(6.0\,\text{V}) = 1.2 \times 10^{-4}\,\text{C} = 120\,\mu\text{C}.$$

$$\boxed{V_M - V_B = 18\,\text{V}, \qquad u_C = 6.0\,\text{V}, \qquad q_C = 120\,\mu\text{C}}$$

> ✓ **Sanity Check:**
> Checking KVL around the closed upper loop $D \to L \to A \to R_{3a} \to M \to C \to R_c \to D$: the drop from A to M across R_{3a} is $R_{3a}I_3 = 1 \times 6.0 = 6.0\,\text{V}$, which equals $u_C = 6.0\,\text{V}$ because both $u_L = 0$ and $u_{R_c} = 0$.

5) Which electrical state variables of L and C remain continuous immediately after an abrupt switching operation in the network? ★☆☆☆☆

Method *Continuity of Magnetic Flux / Inductor Current and Electric Charge / Capacitor Voltage*

> ***Why we do this (Way of Thinking):***
> Stored magnetic energy $W_L = \frac{1}{2}Li_L^2$ and stored electrostatic energy $W_C = \frac{1}{2}Cu_C^2$ cannot jump discontinuously across a zero time interval $\Delta t \to 0$ unless infinite instantaneous power ($u_L \to \infty$ or $i_C \to \infty$) is supplied. Bounded circuit voltages and currents therefore force the derivative variables $i_L(t)$ and $u_C(t)$ to be continuous.

Execution: Across any abrupt switching instant t_0 in a physical circuit with bounded voltages and currents:

- For the ideal coil L ($u_L = L\,di_L/dt$ finite), the **inductor current** $i_L(t)$ remains continuous: $i_L(t_0^+) = i_L(t_0^-)$ (here $i_L(t_0^-) = I_{\text{tot}} = 12.0\,\text{A}$).
- For the ideal capacitor C ($i_C = C\,du_C/dt$ finite), the **capacitor voltage** $u_C(t)$ (and equivalently its stored charge $q_C(t) = Cu_C(t)$) remains continuous: $u_C(t_0^+) = u_C(t_0^-)$ (here $u_C(t_0^-) = 6.0\,\text{V}$).

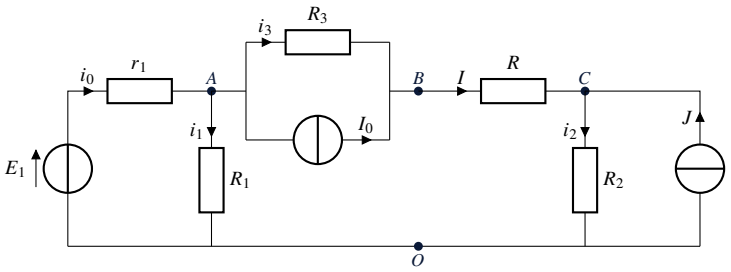
$$\boxed{\begin{aligned} &i_L(t_0^+) = i_L(t_0^-) \quad \text{(inductor current)},\\ &u_C(t_0^+) = u_C(t_0^-),\ \ q_C(t_0^+) = q_C(t_0^-) \quad \text{(capacitor voltage \& charge)}\end{aligned}}$$

> ► **Common Pitfall:**
> By contrast, the dual non-state variables — the inductor voltage $u_L(t)$ and the capacitor current $i_C(t)$ — *can* jump discontinuously at $t = t_0$ whenever switching abruptly alters the surrounding KVL/KCL constraints.

Exercise 4

Part A — Multi-Source DC Network: Circuit (a)

Configuration Summary: *Circuit (a) with $E_1 = 20\,\text{V}$, $r_1 = 4\,\Omega$, $R_1 = 4\,\Omega$, floating parallel bridge ($R_3 = 3\,\Omega \parallel I_0 = 2.0\,\text{A}$) between A and B, central resistor $R = 3\,\Omega$ between B and C, and right-hand parallel pair ($R_2 = 6\,\Omega \parallel J = 1.5\,\text{A}$) between C and reference rail O.*



1) Treating (E_1, r_1) as a single branch, count the principal nodes N, the branches B, the number of independent KCL equations, and the number of independent KVL mesh equations required once the ideal current sources fix their branch currents. ★★☆☆☆

Method ***Topological Graph Counting ($N - 1$ Independent KCL; $(B - N + 1) - N_I$ KVL)***

> ***Why we do this (Way of Thinking):***
> Counting topological invariants before writing any equation prevents both redundant equations and an under-determined system. Every ideal current source fixes its own branch current directly, removing one unknown current while introducing an unknown terminal voltage across the current source; consequently, each ideal current source reduces the number of required KVL equations by one, provided the remaining KVL loops avoid traversing any ideal current source.

Execution:

- **Principal nodes (N):** Junctions connecting three or more branches are O (entire bottom rail), A, B, and C, giving
$$N = 4.$$
- **Branches (B):** Listing each two-terminal path between principal nodes with (E_1, r_1) counted as one branch: (1) (E_1, r_1) between O and A; (2) R_1 between A and O; (3) R_3 between A and B; (4) I_0 between A and B; (5) R between B and C; (6) R_2 between C and O; (7) J between O and C, giving
$$B = 7.$$
- **Independent KCL equations:**
$$N_{\text{KCL}} = N - 1 = 4 - 1 = 3.$$
- **Required KVL mesh equations:** The graph has $M = B - N + 1 = 7 - 4 + 1 = 4$ fundamental meshes. Since $N_I = 2$ branches contain ideal current sources ($I_0 = 2.0\,\text{A}$ and $J = 1.5\,\text{A}$), only $B - N_I = 7 - 2 = 5$ branch currents are unknown, of which $N_{\text{KCL}} = 3$ are eliminated by KCL, leaving
$$N_{\text{KVL}} = (B - N + 1) - N_I = 4 - 2 = 2$$
independent KVL mesh equations (written on loops avoiding both I_0 and J).

$$N = 4, \qquad B = 7, \qquad N_{\text{KCL}} = N - 1 = 3, \qquad N_{\text{KVL}} = (B - N + 1) - 2 = 2$$

> **► Common Pitfall:**
> Writing KVL around a mesh containing I_0 or J while setting the voltage across the ideal current source to zero is a severe conceptual error: an ideal current source sustains whatever nonzero terminal voltage (U_{I_0} or U_J) the external network dictates.

2) Find the current I through resistor R using generator equivalences. ★★★☆☆

Method ***Iterative Thévenin–Norton Reduction Preserving the Target Branch R***

> ***Why we do this (Way of Thinking):***
> To find I through R via generator equivalences, resistor R (between B and C) must remain untouched while the active sub-network left of B and the active sub-network right of C are folded step by step into two single Thévenin branches (E_L, R_L) and (E_R, R_R) sharing rail O. Notice in particular that the floating parallel pair $(R_3 \parallel I_0)$ between A and B is itself a local Norton dipole that converts directly into a series Thévenin branch $(E_0 = R_3 I_0,\ R_3)$.

Execution:

Step 1 — Reduce the sub-network left of node A (between A and O): Convert $(E_1 = 20\,\text{V},\ r_1 = 4\,\Omega)$ to its Norton form $(\eta_1,\ r_1)$:

$$\eta_1 = \frac{E_1}{r_1} = \frac{20\,\text{V}}{4\,\Omega} = 5.0\,\text{A} \quad (\text{directed } O \to A,\ \text{in parallel with } r_1 = 4\,\Omega).$$

Merge $r_1 = 4\,\Omega$ in parallel with shunt $R_1 = 4\,\Omega$ across (A, O), then convert back to Thévenin (E_A, R_A):

$$R_A = \frac{r_1 R_1}{r_1 + R_1} = \frac{4 \times 4}{4 + 4} = 2.0\,\Omega, \qquad E_A = R_A \eta_1 = (2.0\,\Omega)(5.0\,\text{A}) = 10.0\,\text{V} \quad (\text{arrow } O \to A).$$

Step 2 — Convert the floating bridge $(R_3 \parallel I_0)$ and fold to node B: Between A and B, ideal current source $I_0 = 2.0\,\text{A}$ (directed $A \to B$) in parallel with $R_3 = 3\,\Omega$ is equivalent to a series Thévenin generator (E_0, R_3) whose voltage arrow points in the direction of I_0 ($A \to B$, raising potential from A to B):

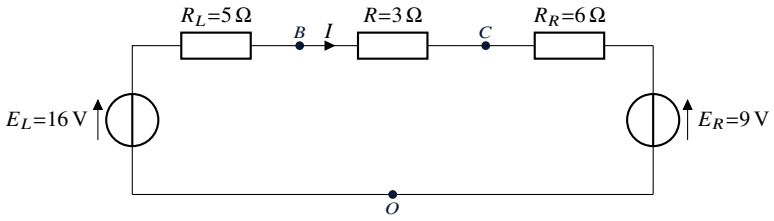
$$E_0 = R_3\, I_0 = (3\,\Omega)(2.0\,\text{A}) = 6.0\,\text{V} \quad (\text{arrow } A \to B), \qquad R_3 = 3.0\,\Omega.$$

Adding (E_A, R_A) (from O to A) in series with (E_0, R_3) (from A to B) yields the complete left Thévenin equivalent (E_L, R_L) between B and O:

$$E_L = E_A + E_0 = 10.0 + 6.0 = 16.0\,\text{V} \quad (\text{arrow } O \to B), \qquad R_L = R_A + R_3 = 2.0 + 3.0 = 5.0\,\Omega.$$

Step 3 — Convert the right-hand pair $(R_2 \parallel J)$ between C and O: Current source $J = 1.5\,\text{A}$ (directed $O \to C$) in parallel with $R_2 = 6\,\Omega$ transforms into the right Thévenin equivalent (E_R, R_R):

$$E_R = R_2\, J = (6\,\Omega)(1.5\,\text{A}) = 9.0\,\text{V} \quad (\text{arrow } O \to C), \qquad R_R = R_2 = 6.0\,\Omega.$$



Step 4 — Single-loop KVL for I: Traversing the reduced single loop clockwise ($O \to B \to C \to O$):

$$+E_L - R_L\, I - R\, I - R_R\, I - E_R = 0 \implies I = \frac{E_L - E_R}{R_L + R + R_R} = \frac{16.0 - 9.0}{5.0 + 3.0 + 6.0} = \frac{7.0}{14.0} = 0.50\,\text{A}.$$

$$\boxed{I = 0.50\,\text{A}}$$

> ► **Common Pitfall:**
> Pay close attention to the polarity of the floating Norton-to-Thévenin conversion $(I_0 \parallel R_3)$: because I_0 pumps charge from A toward B, its open-circuit potential satisfies $V_B - V_A = +R_3 I_0 = +6.0\,\text{V}$, so E_0 *aids* E_A ($E_L = 10 + 6 = 16\,\text{V}$, not $10 - 6 = 4\,\text{V}$).

3) Write KCL and KVL directly on Circuit (a) to determine all branch currents (i_0, i_1, i_3, I, i_2), verifying agreement with question 2, and deduce the terminal voltage $U_J = V_C - V_O$ across the ideal current source J. ★★★☆☆

Method ***Direct Kirchhoff Formulation Eliminating Current-Source Meshes***

> ***Why we do this (Way of Thinking):***
> Following the topological count of question 1, we first express (i_3, i_1, i_2) in terms of just two unknowns (i_0, I) using the $N_{\text{KCL}} = 3$ node equations at B, A, and C, and then write $N_{\text{KVL}} = 2$ mesh equations along purely resistive/voltage-source paths (passing through R_3 rather than I_0, and through R_2 rather than J).

Execution:

Step 1 — Three independent KCL equations at nodes B, A, and C:

$$\begin{aligned}
\text{Node } B: &\quad i_3 + I_0 = I \implies i_3 = I - I_0 = I - 2.0,\\
\text{Node } A: &\quad i_0 = i_1 + i_3 + I_0 = i_1 + I \implies i_1 = i_0 - I,\\
\text{Node } C: &\quad I + J = i_2 \implies i_2 = I + J = I + 1.5.
\end{aligned}$$

Step 2 — Two independent KVL mesh equations avoiding I_0 and J:

- *Mesh 1 (clockwise $O \to E_1 \to r_1 \to A \to R_1 \to O$):*

$$E_1 - r_1\, i_0 - R_1\, i_1 = 0 \implies 20 - 4\, i_0 - 4(i_0 - I) = 0 \implies 20 - 8\, i_0 + 4\, I = 0 \implies i_0 = \frac{5 + I}{2}.$$

- *Mesh 2 (clockwise $O \to R_1 \to A \to R_3 \to B \to R \to C \to R_2 \to O$, ascending R_1 opposite to i_1):*

$$+R_1\, i_1 - R_3\, i_3 - R\, I - R_2\, i_2 = 0.$$

Sub **Substitute $i_1 = i_0 - I$, $i_3 = I - 2.0$, and $i_2 = I + 1.5$ into Mesh 2:**

$$\begin{aligned}
4(i_0 - I) - 3(I - 2.0) - 3\, I - 6(I + 1.5) &= 0\\
4\, i_0 - 4\, I - 3\, I + 6.0 - 3\, I - 6\, I - 9.0 &= 0\\
4\, i_0 - 16\, I - 3.0 &= 0.
\end{aligned}$$

Sub **Substitute $4\, i_0 = 2(5 + I) = 10 + 2\, I$ from Mesh 1:**

$$(10 + 2\, I) - 16\, I - 3.0 = 0 \implies 7.0 - 14\, I = 0 \implies I = \frac{7.0}{14} = 0.50\,\text{A},$$

in exact agreement with question 2.

Step 3 — Remaining branch currents and terminal voltage U_J:

$$\begin{aligned}
i_0 &= \frac{5 + 0.50}{2} = 2.75\,\text{A}, \qquad i_1 = i_0 - I = 2.75 - 0.50 = 2.25\,\text{A},\\
i_3 &= I - 2.0 = 0.50 - 2.0 = -1.50\,\text{A}, \qquad i_2 = I + 1.5 = 0.50 + 1.5 = 2.00\,\text{A}.
\end{aligned}$$

Because ideal current source J lies directly in parallel with R_2 between C and O:

$$U_J = V_C - V_O = R_2\, i_2 = (6\,\Omega)(2.00\,\text{A}) = 12.0\,\text{V}.$$

$$\boxed{i_0 = 2.75\,\text{A},\ i_1 = 2.25\,\text{A},\ i_3 = -1.50\,\text{A},\ I = 0.50\,\text{A},\ i_2 = 2.00\,\text{A},\ U_J = 12.0\,\text{V}}$$

> **✓ Sanity Check:**
> Cross-check node potentials relative to $V_O = 0\,\text{V}$: $V_A = R_1 i_1 = 4 \times 2.25 = 9.0\,\text{V}$ (also $E_1 - r_1 i_0 = 20 - 4 \times 2.75 = 9.0\,\text{V}$); $V_B = V_A - R_3 i_3 = 9.0 - 3(-1.50) = 13.5\,\text{V}$; $V_C = V_B - RI = 13.5 - 3(0.50) = 12.0\,\text{V} = U_J$. Every branch satisfies Ohm's law and KVL simultaneously.

4) Without re-solving the full system, state by how much the KCL current balance at node C shifts if the physical orientation of source J (1.5 A) is reversed while I is held fixed. ★☆☆☆☆

Method ***Algebraic Sign Flip of a Current Source at a Node ($+J \longrightarrow -J$, Shift $2J$)***

> ***Why we do this (Way of Thinking):***
> Reversing an arrow from entering node C ($+J$) to leaving node C ($-J$) does not merely remove J; it subtracts J after removing the original $+J$, producing a net algebraic swing of $-2J$.

Execution: With J directed upward into node C, KCL at C reads $i_2 = I + J$. When J is reversed so it draws 1.5 A downward out of node C, KCL at C becomes $i_2^{\text{new}} + J = I \implies i_2^{\text{new}} = I - J$. At fixed I:

$$\Delta i_2 = i_2^{\text{new}} - i_2 = (I - J) - (I + J) = -2\,J = -2(1.5\,\text{A}) = -3.0\,\text{A},$$

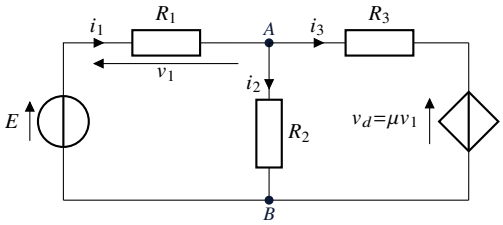
shifting i_2 by a magnitude of $2\,J = 3.0$ A (from +2.00 A to $0.50 - 1.50 = -1.00$ A at $I = 0.50$ A).

$$\Delta i_2 = -2\,J = -3.0\,\text{A} \quad (\text{magnitude shift } |\Delta i_2| = 2\,J = 3.0\,\text{A})$$

> ► **Common Pitfall:**
> Students frequently answer $|\Delta i_2| = J = 1.5$ A (which corresponds to turning J *off*, $J \to 0$) instead of $|\Delta i_2| = 2J = 3.0$ A (which corresponds to *reversing* J, $+J \to -J$).

Part B — Network with a Dependent Source: Circuit (b)

Configuration Summary: *Circuit (b) with $E = 18$ V, $R_1 = 3\,\Omega$, $R_2 = 4\,\Omega$, $R_3 = 6\,\Omega$, and diamond controlled source $v_d = \mu\, v_1$ ($\mu = 3$, $v_1 = R_1\, i_1$).*



5) Name the exact category of the controlled source $v_d = \mu\, v_1$ and give the physical dimension of μ. ★☆☆☆☆

Method ***Four-Type Controlled Source Classification and Dimensional Homogeneity***

> ***Why we do this (Way of Thinking):***
> A diamond source with a voltage arrow inside (or voltage law $v_d = \dots$) is a controlled *voltage* source; inspecting whether its controlling variable is a voltage (v_1) or a current fixes the prefix and the physical unit of its gain coefficient.

Execution: Because the source imposes an output **voltage** v_d proportional to a controlling **voltage** v_1, it is a **Voltage-Controlled Voltage Source (VCVS)**. By dimensional homogeneity of $v_d = \mu\, v_1$:

$$[\mu] = \frac{[v_d]}{[v_1]} = \frac{\text{V}}{\text{V}} = 1 \quad (\text{dimensionless voltage amplification factor}).$$

Category: Voltage-Controlled Voltage Source (VCVS), $[\mu] = 1$ (dimensionless)

> ✓ **Sanity Check:**
> By contrast, a Current-Controlled Voltage Source ($v_d = r_m i_x$) has a transresistance gain in Ω, a Voltage-Controlled Current Source ($i_d = g_m v_x$) has a transconductance gain in S, and a Current-Controlled Current Source ($i_d = \beta i_x$) is dimensionless.

6) Apply KCL and KVL to determine (i_1, i_2, i_3), the controlled voltage v_d, the power absorbed or supplied by the controlled source, and the equivalent input resistance $R_{\text{in}} = E/i_1$ seen by source E. ★★★☆☆

Method ***Coupled Mesh Analysis Incorporating the Controlling-Variable Constraint*** $v_1 = R_1\, i_1$

> ***Why we do this (Way of Thinking):***
> When analyzing a network containing a dependent source, never deactivate or detach the controlling variable: express the dependent quantity $v_d = \mu v_1 = \mu R_1 i_1$ directly in terms of the unknown mesh/branch currents at the very start, then solve the resulting coupled linear KCL/KVL equations. Because a dependent source can either inject or absorb energy depending on feedback polarity, its influence shifts the effective input resistance $R_{\text{in}} = E/i_1$ seen by the driving source E.

Execution:

Step 1 — Express v_d and i_3 in terms of (i_1, i_2): From Ohm's law on $R_1 = 3\,\Omega$ and $\mu = 3$:

$$v_1 = R_1\, i_1 = 3\, i_1 \implies v_d = \mu\, v_1 = 3(3\, i_1) = 9\, i_1.$$

By KCL at node A:

$$i_1 = i_2 + i_3 \implies i_3 = i_1 - i_2.$$

Step 2 — Two independent KVL mesh equations:

- *Left mesh (clockwise $B \to E \to R_1 \to A \to R_2 \to B$):*

$$E - R_1\, i_1 - R_2\, i_2 = 0 \implies 18 - 3\, i_1 - 4\, i_2 = 0 \implies 4\, i_2 = 18 - 3\, i_1.$$

- *Right mesh (clockwise $B \to R_2 \to A \to R_3 \to v_d \to B$, ascending R_2 against i_2 and descending v_d against its upward voltage arrow):*

$$+R_2\, i_2 - R_3\, i_3 - v_d = 0.$$

Sub **Substitute $i_3 = i_1 - i_2$ and $v_d = 9\, i_1$ into the right-mesh equation:**

$$4\, i_2 - 6(i_1 - i_2) - 9\, i_1 = 0 \implies 4\, i_2 - 6\, i_1 + 6\, i_2 - 9\, i_1 = 0 \implies 10\, i_2 = 15\, i_1 \implies i_2 = 1.5\, i_1.$$

Sub **Substitute $i_2 = 1.5\, i_1$ into the left-mesh equation $4\, i_2 = 18 - 3\, i_1$:**

$$4(1.5\, i_1) = 18 - 3\, i_1 \implies 6\, i_1 + 3\, i_1 = 18 \implies 9\, i_1 = 18 \implies i_1 = 2.0\,\text{A}.$$

Hence:

$$i_2 = 1.5 \times 2.0 = 3.0\,\text{A}, \qquad i_3 = i_1 - i_2 = 2.0 - 3.0 = -1.0\,\text{A}, \qquad v_d = 9 \times 2.0 = 18.0\,\text{V}.$$

Step 3 — Power of the controlled source and input resistance R_{in}: Current i_3 flows rightward through R_3 and downward into the upper terminal (arrowhead of v_d) of the controlled source, placing (v_d, i_3) in the **receiver convention**:

$$p_{d,\text{rec}} = v_d\, i_3 = (18.0\,\text{V})(-1.0\,\text{A}) = -18.0\,\text{W} < 0 \implies \text{the controlled source } \textbf{supplies } 18.0\,\text{W}.$$

The equivalent input resistance seen by independent source E is:

$$R_{\text{in}} = \frac{E}{i_1} = \frac{18\,\text{V}}{2.0\,\text{A}} = 9.0\,\Omega.$$

$i_1 = 2.0\,\text{A},\ i_2 = 3.0\,\text{A},\ i_3 = -1.0\,\text{A},\ v_d = 18.0\,\text{V},$ VCVS supplies $18.0\,\text{W},\ R_{\text{in}} = 9.0\,\Omega$

> ✓ **Sanity Check:**
> Global power conservation check: independent source E supplies $E\, i_1 = 18 \times 2.0 = 36.0\,\text{W}$ and the VCVS supplies $18.0\,\text{W}$ (total supplied = $54.0\,\text{W}$), while the three resistors dissipate $R_1 i_1^2 + R_2 i_2^2 + R_3 i_3^2 = 3(4) + 4(9) + 6(1) = 12 + 36 + 6 = 54.0\,\text{W}$. Notice also that without the active feedback ($\mu = 0$), the passive resistance would be $R_1 + (R_2 \parallel R_3) = 3 + 2.4 = 5.4\,\Omega$; because the VCVS pushes current back toward node A ($i_3 = -1.0\,\text{A}$), it opposes i_1 and raises R_{in} to $9.0\,\Omega$.

Exercise 5

Configuration Summary: *Sinusoidal steady-state dipole with* $u(t) = 120\sqrt{2}\sin(100\pi t - \pi/4)$ V *and* $i(t) = 6\sqrt{2}\cos(100\pi t - 5\pi/12)$ A *(t in seconds).*

1) Determine the angular frequency ω, frequency f, and period T. Rewrite $i(t)$ in the standard $\sin(\omega t + \varphi_i)$ reference, and express both initial phases φ_u and φ_i in radians and in degrees. ★☆☆☆☆

Method ***Harmonic Parameter Identification and Quadrature Identity*** $\cos\alpha = \sin(\alpha + \pi/2)$

> ***Why we do this (Way of Thinking):***
> Comparing two sinusoidal signals requires expressing both waveforms in the *exact same trigonometric reference* ($\sin(\omega t + \varphi)$ with positive amplitude). Because $u(t)$ is already given as a sine wave while $i(t)$ is given as a cosine wave, applying $\cos\alpha = \sin(\alpha + \pi/2)$ shifts the cosine argument by $+\pi/2$ before extracting φ_i.

Execution:

Step 1 — Angular frequency ω, frequency f, and period T: By inspection of $\omega t = 100\pi t$:

$$\omega = 100\pi \text{ rad s}^{-1} \approx 314.16 \text{ rad s}^{-1}, \qquad f = \frac{\omega}{2\pi} = \frac{100\pi}{2\pi} = 50 \text{ Hz}, \qquad T = \frac{1}{f} = \frac{1}{50} = 0.020 \text{ s} = 20 \text{ ms}.$$

Step 2 — Conversion of $i(t)$ to the sine reference and initial phases:

Sub **Apply $\cos\alpha = \sin(\alpha + \pi/2)$ with $\alpha = 100\pi t - 5\pi/12$:**

$$i(t) = 6\sqrt{2}\sin\left(100\pi t - \frac{5\pi}{12} + \frac{\pi}{2}\right) = 6\sqrt{2}\sin\left(100\pi t + \frac{-5\pi + 6\pi}{12}\right) = 6\sqrt{2}\sin\left(100\pi t + \frac{\pi}{12}\right) \text{ A}.$$

Converting φ_u and φ_i between radians and degrees via $\varphi_{\text{deg}} = \varphi_{\text{rad}} \times \dfrac{180^\circ}{\pi}$:

$$\varphi_u = -\frac{\pi}{4} \text{ rad} = -45^\circ, \qquad \varphi_i = +\frac{\pi}{12} \text{ rad} = +15^\circ.$$

$$\omega = 100\pi \text{ rad s}^{-1}, \quad f = 50 \text{ Hz}, \quad T = 20 \text{ ms}, \quad \varphi_u = -\frac{\pi}{4} \text{ rad} = -45^\circ, \quad \varphi_i = +\frac{\pi}{12} \text{ rad} = +15^\circ$$

> ► **Common Pitfall:**
> Beware of subtracting $\pi/2$ instead of adding $\pi/2$: $\cos(0) = +1 = \sin(+\pi/2)$, whereas $\sin(-\pi/2) = -1$. Subtracting $\pi/2$ introduces a 180° (π rad) polarity error in $\bar{I}$.

2) By explicit integration over one period T, evaluate the full-period mean voltage $\langle u \rangle$ and establish the RMS value U of $u(t)$. ★★☆☆☆

Method ***Full-Period Definite Integration and Trigonometric Linearization*** $\sin^2\theta = (1 - \cos 2\theta)/2$

> ***Why we do this (Way of Thinking):***
> Evaluating $\langle u \rangle = \frac{1}{T}\int_0^T u(t)\,dt$ and $U = \sqrt{\frac{1}{T}\int_0^T u^2(t)\,dt}$ from first principles hinges on changing variable to the phase angle $\theta = \omega t + \varphi_u$, which maps any full time period $[0, T]$ (since $\omega T = 2\pi$) onto a full 2π-radian interval $[\varphi_u, \varphi_u + 2\pi]$ over which every pure harmonic $\sin(n\theta)$ or $\cos(n\theta)$ ($n \in \mathbb{N}^*$) integrates to zero.

Execution: Write $u(t) = U_m \sin(\omega t + \varphi_u)$ with $U_m = 120\sqrt{2}$ V, $\varphi_u = -\pi/4$, and $\omega T = 2\pi$.

Step 1 — Full-period mean voltage $\langle u \rangle$:

$$\langle u \rangle = \frac{1}{T}\int_0^T U_m \sin(\omega t + \varphi_u)\,dt.$$

Sub **Let $\theta = \omega t + \varphi_u$, so $d\theta = \omega\, dt$ ($dt = d\theta/\omega$); bounds: $t = 0 \implies \theta = \varphi_u$ and $t = T \implies \theta = \varphi_u + \omega T = \varphi_u + 2\pi$.**

$$\langle u \rangle = \frac{U_m}{\omega T}\int_{\varphi_u}^{\varphi_u+2\pi} \sin\theta\, d\theta = \frac{U_m}{2\pi}\Big[-\cos\theta\Big]_{\varphi_u}^{\varphi_u+2\pi} = \frac{U_m}{2\pi}\big[-\cos(\varphi_u+2\pi)+\cos\varphi_u\big] = \frac{U_m}{2\pi}(0) = 0\,\text{V}.$$

Step 2 — Mean-square integral U^2 and RMS voltage U:

$$U^2 = \frac{1}{T}\int_0^T u^2(t)\, dt = \frac{U_m^2}{T}\int_0^T \sin^2(\omega t+\varphi_u)\, dt.$$

Using the same phase substitution $\theta = \omega t + \varphi_u$ ($dt = d\theta/\omega$, $\omega T = 2\pi$) together with the half-angle identity $\sin^2\theta = \dfrac{1-\cos(2\theta)}{2}$:

$$\begin{aligned} U^2 &= \frac{U_m^2}{2\pi}\int_{\varphi_u}^{\varphi_u+2\pi}\frac{1-\cos(2\theta)}{2}\, d\theta = \frac{U_m^2}{4\pi}\left[\theta - \frac{\sin(2\theta)}{2}\right]_{\varphi_u}^{\varphi_u+2\pi} \\ &= \frac{U_m^2}{4\pi}\left[(\varphi_u+2\pi-\varphi_u) - \frac{\sin(2\varphi_u+4\pi)-\sin(2\varphi_u)}{2}\right] = \frac{U_m^2}{4\pi}(2\pi-0) = \frac{U_m^2}{2}. \end{aligned}$$

Taking the positive square root with $U_m = 120\sqrt{2}\,\text{V}$:

$$U = \sqrt{U^2} = \frac{U_m}{\sqrt{2}} = \frac{120\sqrt{2}}{\sqrt{2}} = 120\,\text{V}.$$

$$\boxed{\langle u \rangle = 0\,\text{V}, \qquad U = \frac{U_m}{\sqrt{2}} = 120\,\text{V}}$$

✓ **Sanity Check:**
Notice that the initial phase $\varphi_u = -\pi/4$ drops out completely of both full-period integrals because shifting a periodic waveform horizontally along the time axis never alters the signed or squared area enclosed over one full cycle T.

3) Within the first positive period $t \in [0, T]$, calculate the exact instants of the rising zero-crossing t_{0+}, maximum $t_{\max}$, falling zero-crossing t_{0-}, and minimum $t_{\min}$ of $u(t)$. ★★☆☆☆

Method ***Phase-Argument Inversion on the Interval*** $\theta(t) \in [\varphi_u,\ \varphi_u + 2\pi]$

Why we do this (Way of Thinking):
As t sweeps across the first period [0, 20 ms], the instantaneous phase $\theta(t) = 100\pi t - \pi/4$ sweeps monotonically across $[-\pi/4,\ 7\pi/4]$. Within $[-\pi/4,\ 7\pi/4]$, $\sin\theta$ hits its rising zero at $\theta = 0$, its peak $+1$ at $\theta = \pi/2$, its falling zero at $\theta = \pi$, and its trough -1 at $\theta = 3\pi/2$.

Execution: Solving $\theta(t) = 100\pi t - \dfrac{\pi}{4} = \theta_\star$ for each of the four canonical angles $\theta_\star \in \left\{0,\ \dfrac{\pi}{2},\ \pi,\ \dfrac{3\pi}{2}\right\} \subset \left[-\dfrac{\pi}{4},\ \dfrac{7\pi}{4}\right]$:

$$\begin{aligned} \text{Rising zero } (\theta = 0): &\quad 100\pi t_{0+} - \frac{\pi}{4} = 0 \implies t_{0+} = \frac{1}{400}\,\text{s} = 2.5\,\text{ms}, \\ \text{Maximum } (\theta = \pi/2): &\quad 100\pi t_{\max} - \frac{\pi}{4} = \frac{\pi}{2} \implies 100\pi t_{\max} = \frac{3\pi}{4} \implies t_{\max} = \frac{3}{400}\,\text{s} = 7.5\,\text{ms}, \\ \text{Falling zero } (\theta = \pi): &\quad 100\pi t_{0-} - \frac{\pi}{4} = \pi \implies 100\pi t_{0-} = \frac{5\pi}{4} \implies t_{0-} = \frac{5}{400}\,\text{s} = 12.5\,\text{ms}, \\ \text{Minimum } (\theta = 3\pi/2): &\quad 100\pi t_{\min} - \frac{\pi}{4} = \frac{3\pi}{2} \implies 100\pi t_{\min} = \frac{7\pi}{4} \implies t_{\min} = \frac{7}{400}\,\text{s} = 17.5\,\text{ms}. \end{aligned}$$

$$t_{0+} = 2.5\,\text{ms}, \qquad t_{\max} = 7.5\,\text{ms}, \qquad t_{0-} = 12.5\,\text{ms}, \qquad t_{\min} = 17.5\,\text{ms}$$

✓ **Sanity Check:**
All four instants lie strictly inside $[0,\ 20\,\text{ms}]$ and are separated consecutively by exactly one quarter-period $T/4 = 20\,\text{ms}/4 = 5.0\,\text{ms}$: $2.5 \to 7.5 \to 12.5 \to 17.5\,\text{ms}$.

4) Write the RMS complex phasors $\bar{U}$ and $\bar{I}$, and specify the lengths (in cm) and polar angles of the Fresnel vectors $\vec{U}$ and $\vec{I}$ using scales $30\,\text{V cm}^{-1}$ and $2\,\text{A cm}^{-1}$. ★☆☆☆☆

Method *RMS Complex Phasor Mapping and Linear Fresnel Scale Division*

Why we do this (Way of Thinking):
In power and circuit engineering, complex phasors $\bar{U} = Ue^{j\varphi_u}$ and Fresnel vectors $\vec{U}(U, \varphi_u)$ are normalized by the *RMS* magnitudes $U = U_m/\sqrt{2}$ and $I = I_m/\sqrt{2}$ (not the peak values U_m, I_m) so that active and apparent powers write directly as $P = UI\cos\varphi$ and $S = UI$ without extraneous factors of $1/2$.

Execution: With RMS values $U = 120\,\text{V}$ and $I = I_m/\sqrt{2} = 6\sqrt{2}/\sqrt{2} = 6.0\,\text{A}$, and initial phases $\varphi_u = -\pi/4 = -45°$, $\varphi_i = +\pi/12 = +15°$:

$$\bar{U} = 120\,e^{-j\pi/4} = 120\angle{-45°} = 120\left(\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}\right) = 60\sqrt{2}(1-j)\ \text{V} \approx (84.85 - j\,84.85)\ \text{V},$$

$$\bar{I} = 6.0\,e^{+j\pi/12} = 6.0\angle{+15°} = 6.0\big(\cos 15° + j\sin 15°\big) \approx (5.80 + j\,1.55)\ \text{A}.$$

Dividing each RMS modulus by its prescribed graphical scale gives the drawn Fresnel vector lengths:

$$\ell_U = \frac{120\,\text{V}}{30\,\text{V cm}^{-1}} = 4.0\,\text{cm at } \varphi_u = -45°, \qquad \ell_I = \frac{6.0\,\text{A}}{2\,\text{A cm}^{-1}} = 3.0\,\text{cm at } \varphi_i = +15°.$$

$$\bar{U} = 120\angle{-45°}\,\text{V},\ \ \bar{I} = 6.0\angle{+15°}\,\text{A},\ \ \vec{U}: 4.0\,\text{cm at } -45°,\ \ \vec{I}: 3.0\,\text{cm at } +15°$$

► **Common Pitfall:**
Do not divide $U_m = 120\sqrt{2}\,\text{V}$ by $30\,\text{V cm}^{-1}$ ($\ell_U \neq 4\sqrt{2}\,\text{cm}$): standard RMS Fresnel conventions scale RMS magnitudes U and I directly.

5) Calculate the phase shift $\varphi = \varphi_u - \varphi_i$, state which waveform leads the other, and deduce the corresponding time shift $|\Delta t|$ between their rising zero-crossings. ★★☆☆☆

Method *Impedance Phase Angle $\varphi = \varphi_u - \varphi_i$ and Proportional Time Offset $|\Delta t| = |\varphi|/\omega$*

Why we do this (Way of Thinking):
By convention, the dipole phase angle is defined as $\varphi = \varphi_u - \varphi_i = \arg(\bar{Z})$. If $\varphi > 0$, voltage leads current (inductive character); if $\varphi < 0$, voltage lags current, meaning current leads voltage (capacitive character). Converting angular shift $|\varphi|$ (in radians) to time shift divides by ω because one full period T corresponds to 2π radians.

Execution: Subtracting φ_i from φ_u:

$$\varphi = \varphi_u - \varphi_i = -\frac{\pi}{4} - \frac{\pi}{12} = \frac{-3\pi - \pi}{12} = -\frac{4\pi}{12} = -\frac{\pi}{3}\ \text{rad} = -60°.$$

Since $\varphi = \varphi_u - \varphi_i < 0$ ($\varphi_i > \varphi_u$), **the current $i(t)$ leads the voltage $u(t)$ by** $60°$ **(**$\pi/3$ rad**)** (equivalently, $u(t)$ lags $i(t)$ by $60°$, indicating a capacitive dipole). The time separation between their rising zero-crossings is:

$$|\Delta t| = \frac{|\varphi|}{\omega} = \frac{\pi/3}{100\pi} = \frac{1}{300}\ \text{s} = \frac{10}{3}\ \text{ms} \approx 3.33\,\text{ms}.$$

$$\varphi = -\frac{\pi}{3}\,\text{rad} = -60^\circ \text{ (current } i(t) \text{ leads } u(t)\text{)}, \qquad |\Delta t| = \frac{10}{3}\,\text{ms} \approx 3.33\,\text{ms}$$

✓ **Sanity Check:**
Check directly from the rising zero-crossing times: $i(t) = 6\sqrt{2}\sin(100\pi t + \pi/12)$ crosses zero with rising slope at $100\pi t_{0+,i} + \pi/12 = 0 \implies t_{0+,i} = -\frac{1}{1200}\,\text{s} = -\frac{5}{6}\,\text{ms}$, which occurs *earlier* than $t_{0+,u} = +2.5\,\text{ms} = +\frac{15}{6}\,\text{ms}$ by $t_{0+,u} - t_{0+,i} = \frac{15}{6} - \left(-\frac{5}{6}\right) = \frac{20}{6} = \frac{10}{3}\,\text{ms}$.

Exercise 6

Configuration Summary: *Unknown series dipole $\mathcal{D} = (r_0, X)$ in series with current-sensing resistor $R = 80\,\Omega$ across generator $u(t)$. Oscilloscope settings: $S_{V1} = 4\,\text{V/div}$ (CH1: $u(t)$), $S_{V2} = 2\,\text{V/div}$ (CH2: $u_R(t)$), $S_t = 1\,\text{ms/div}$. Screen readings: $N_T = 6.0$ div, $N_{y1} = 2.5$ div, $N_{y2} = 2.0$ div, rising zero-crossing of CH2 at 0.5 div, rising zero-crossing of CH1 at 1.5 div.*

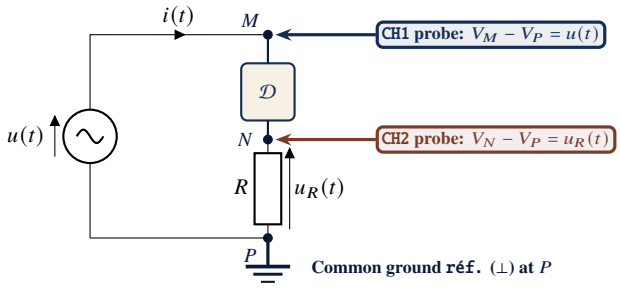
1) Redraw Figure (a) and indicate at which nodes (M, N, or P) the CH1 probe, the CH2 probe, and the common oscilloscope reference ground (réf. / ⊥) must be connected. ★★☆☆☆

Method *Single-Reference Dual-Channel Oscilloscope Probe Topology*

Why we do this (Way of Thinking):
An oscilloscope measures exclusively node potentials referenced to a *single* shared ground node (réf. / ⊥). To observe both $u(t) = V_M - V_P$ and $u_R(t) = V_N - V_P$ simultaneously with positive polarity, both target voltages must share node P as their minus terminal; hence node P receives the common ground clip, node M receives CH1, and node N receives CH2.

Execution:

- **Common reference ground (réf. / ⊥):** Connected to **node** P ($V_P = 0\,\text{V}$).
- **Channel 1 probe (CH1):** Connected to **node** M, measuring $V_M - V_P = u(t)$.
- **Channel 2 probe (CH2):** Connected to **node** N, measuring $V_N - V_P = u_R(t) = R\,i(t)$.



CH1 ⟶ node M, CH2 ⟶ node N, réf. (⊥) ⟶ node P

✓ **Sanity Check:**
Because $R = 80\,\Omega$ is purely resistive, $u_R(t) = R\,i(t)$ has zero phase shift relative to $i(t)$ ($\varphi_{u_R} = \varphi_i$), allowing the voltage-only oscilloscope to display the exact phase of $i(t)$ on CH2.

2) Explain what happens if an operator connects the CH1 ground clip to node P and the CH2 ground clip to node N. If $\mathcal{D}$ and R were physically interchanged in Figure (a), what oscilloscope channel function

must be activated to display $u_R(t)$ with the proper sign? ★★☆☆☆

Method *Internal Chassis-Ground Bonding Analysis and Channel Polarity Inversion (`INV`)*

> ***Why we do this (Way of Thinking):***
> Inside a standard dual-trace oscilloscope, the outer metallic shields (ground clips) of all BNC inputs are permanently wired together to the instrument chassis. Connecting two ground clips to two distinct circuit nodes forces those two nodes to the same potential through a low-resistance metallic path.

Execution:

(a) **Consequence of clipping `CH1` ground to P and `CH2` ground to N:** Because both ground clips are internally bonded together inside the oscilloscope chassis, connecting them simultaneously to P and N places a direct metallic wire across nodes N and P, **short-circuiting resistor** R ($u_R(t) = 0\,\text{V}$ on `CH2`, altering the circuit impedance to $\mathcal{D}$ alone and risking excessive fault current through the ground lead).

(b) **Interchanging $\mathcal{D}$ and R:** If R occupies the upper branch (M–N) while $\mathcal{D}$ occupies the lower branch (N–P), placing the common ground at M so `CH2` at N spans R yields $V_N - V_M = -R\,i(t) = -u_R(t)$ (inverted polarity, shifted by 180°). One must therefore activate the **channel inversion function (`INV` / Invert)** on `CH2` to multiply $-u_R(t)$ by -1 and recover $+u_R(t)$ (or use the differential `MATH` subtraction `CH1` − `CH2` if ground is kept at P).

> Dual grounds at P and $N \Longrightarrow$ short-circuits resistor R; swapped $(\mathcal{D}, R) \Longrightarrow$ activate `INV` (Invert)

> **► Common Pitfall:**
> Forgetting to press `INV` when probing $-u_R(t)$ flips the apparent current trace upside down, introducing a spurious $\pm 180°$ (π rad) error in the measured phase angle φ and falsely turning a valid passive $\cos\varphi > 0$ reading into $\cos(\varphi \pm \pi) < 0$.

3) From Figure (b), determine the period T, frequency f, angular frequency ω, peak voltages U_m and U_{Rm}, RMS voltages U and U_R, and the RMS circuit current I. ★★☆☆☆

Method *Oscillogram Division Scaling ($T = N_T S_t$, $U_m = N_y S_V$) and RMS Conversion*

> ***Why we do this (Way of Thinking):***
> Each trace on Figure (b) must be multiplied by its *own* vertical sensitivity ($S_{V1} = 4\,\text{V/div}$ for `CH1` vs. $S_{V2} = 2\,\text{V/div}$ for `CH2`), whereas both channels share the same horizontal time-base scale $S_t = 1\,\text{ms/div}$.

Execution:

Step 1 — Period T, frequency f, and angular frequency ω: One complete cycle spans $N_T = 6.5 - 0.5 = 6.0$ div horizontally (for both `CH2` from 0.5 to 6.5 div and `CH1` from 1.5 to 7.5 div):

$$T = N_T \cdot S_t = 6.0 \times 1\,\text{ms} = 6.0\,\text{ms} = 6.0 \times 10^{-3}\,\text{s},$$
$$f = \frac{1}{T} = \frac{1}{6.0 \times 10^{-3}\,\text{s}} = \frac{500}{3}\,\text{Hz} \approx 166.7\,\text{Hz},$$
$$\omega = \frac{2\pi}{T} = \frac{2\pi}{6.0 \times 10^{-3}\,\text{s}} = \frac{1000\pi}{3}\,\text{rad s}^{-1} \approx 1047.2\,\text{rad s}^{-1}.$$

Step 2 — Peak voltages (U_m, U_{Rm}), RMS voltages (U, U_R), and RMS current I: From the vertical peak deflections $N_{y1} = 2.5$ div (`CH1`) and $N_{y2} = 2.0$ div (`CH2`):

$$U_m = N_{y1} \cdot S_{V1} = 2.5 \times 4\,\text{V} = 10.0\,\text{V} \Longrightarrow U = \frac{U_m}{\sqrt{2}} = \frac{10.0}{\sqrt{2}} = 5\sqrt{2}\,\text{V} \approx 7.07\,\text{V},$$
$$U_{Rm} = N_{y2} \cdot S_{V2} = 2.0 \times 2\,\text{V} = 4.0\,\text{V} \Longrightarrow U_R = \frac{U_{Rm}}{\sqrt{2}} = \frac{4.0}{\sqrt{2}} = 2\sqrt{2}\,\text{V} \approx 2.83\,\text{V}.$$

Applying Ohm's law across $R = 80\,\Omega$ gives the RMS circuit current (and peak current $I_m = U_{Rm}/R = 4.0/80 = 50.0\,\text{mA}$):

$$I = \frac{U_R}{R} = \frac{2\sqrt{2}}{80\,\Omega} = \frac{\sqrt{2}}{40}\,\text{A} \approx 0.035\,36\,\text{A} = 35.36\,\text{mA}.$$

$$T = 6.0\,\text{ms},\quad f \approx 166.7\,\text{Hz},\quad \omega = \frac{1000\pi}{3}\,\text{rad}\,\text{s}^{-1},\quad U = 5\sqrt{2}\,\text{V},\quad U_R = 2\sqrt{2}\,\text{V},\quad I \approx 35.36\,\text{mA}$$

► **Common Pitfall:**
Read the *semi-amplitude* N_y from the central axis 0 V to a crest (2.5 div and 2.0 div), not the peak-to-peak height $2N_y$ (5.0 div and 4.0 div); using peak-to-peak divisions without dividing by 2 doubles U_m, U_{Rm}, and I.

4) Determine the horizontal time offset $|\Delta t|$, the signed phase shift $\varphi = \varphi_u - \varphi_i$ (in radians and degrees), and the power factor $\cos\varphi$ of the complete series circuit. Deduce whether $\mathcal{D}$ is inductive or capacitive.
★★★☆☆

Method ***Time-Axis Zero-Crossing Offset and Earlier-Is-Left Lead/Lag Criterion***

Why we do this (Way of Thinking):
On a horizontal time axis increasing from left to right, the waveform whose rising zero-crossing appears further to the *left* reaches zero at an *earlier* instant t and therefore *leads* in phase. Comparing the rising zero of CH1 (u) against CH2 ($u_R \propto i$) fixes the algebraic sign of $\varphi = \varphi_u - \varphi_i$ unambiguously.

Execution:

Step 1 — Time offset $|\Delta t|$ and phase magnitude $|\varphi|$: CH2 (u_R) crosses zero with rising slope at $x_2 = 0.5$ div, whereas CH1 (u) crosses zero with rising slope at $x_1 = 1.5$ div, giving a horizontal separation $N_{\Delta x} = 1.5 - 0.5 = 1.0$ div:

$$|\Delta t| = N_{\Delta x} \cdot S_t = 1.0 \times 1\,\text{ms} = 1.0\,\text{ms},$$
$$|\varphi| = 2\pi\,\frac{N_{\Delta x}}{N_T} = 2\pi\,\frac{1.0}{6.0} = \frac{\pi}{3}\,\text{rad} = 360^\circ \times \frac{1.0}{6.0} = 60^\circ.$$

Step 2 — Algebraic sign of φ, power factor $\cos\varphi$, and reactive nature of $\mathcal{D}$: Because CH1 ($u(t)$) crosses zero **later** than CH2 ($u_R(t) = R\,i(t)$) — shifted 1.0 div to the **right** — the total voltage $u(t)$ **lags** the circuit current $i(t)$:

$$\varphi = \varphi_u - \varphi_i = -\frac{\pi}{3}\,\text{rad} = -60^\circ < 0, \qquad \cos\varphi = \cos(-60^\circ) = 0.50.$$

Since $\varphi < 0$ (total series reactance $X_{\text{tot}} < 0$) and $R = 80\,\Omega$ is purely resistive, the unknown series dipole $\mathcal{D}$ is **capacitive** (internal resistance r_0 in series with a capacitor C).

$$|\Delta t| = 1.0\,\text{ms},\qquad \varphi = -\frac{\pi}{3}\,\text{rad} = -60^\circ,\qquad \cos\varphi = 0.50,\qquad \mathcal{D}\text{ is capacitive}$$

► **Common Pitfall:**
Do not confuse "larger visually on screen" ($U_m = 2.5$ div $> U_{Rm} = 2.0$ div) or "further to the right" ($x_1 = 1.5 > x_2 = 0.5$) with phase lead: shifting a function $\sin(\omega t + \varphi)$ to the *right* corresponds to $\sin[\omega(t - \Delta t)] = \sin(\omega t - \omega\Delta t)$, which carries a *negative* phase shift (lag).

5) Calculate the modulus $|\bar{Z}_{\text{tot}}|$ of the complete circuit impedance, then deduce the internal resistance r_0 and the value of the reactive component (L or C) inside dipole $\mathcal{D}$. ★★★☆☆

Method ***Complex Impedance Decomposition*** $\bar{Z}_{\text{tot}} = |\bar{Z}_{\text{tot}}|(\cos\varphi + j\sin\varphi) = (R + r_0) - j/(C\omega)$

> ***Why we do this (Way of Thinking):***
> In the series loop, $\bar{Z}_{\text{tot}} = R + \bar{Z}_{\mathcal{D}} = (R + r_0) + jX_{\text{tot}}$. Projecting $|\bar{Z}_{\text{tot}}| = U/I$ onto the real axis ($\cos\varphi$) gives the total circuit resistance $R + r_0$ (from which subtracting $R = 80\,\Omega$ isolates r_0), while projecting onto the imaginary axis ($\sin\varphi$) gives $-1/(C\omega)$.

Execution:

Step 1 — Total impedance modulus $|\bar{Z}_{\text{tot}}|$: Using exact peak or RMS ratios ($U_m = 10.0\,\text{V}$, $I_m = U_{Rm}/R = 4.0/80 = 0.050\,\text{A}$):

$$|\bar{Z}_{\text{tot}}| = \frac{U}{I} = \frac{U_m}{I_m} = \frac{10.0\,\text{V}}{0.050\,\text{A}} = 200\,\Omega.$$

Step 2 — Internal resistance r_0 and capacitance C of dipole $\mathcal{D}$: Equating real parts of $\bar{Z}_{\text{tot}} = (R + r_0) - j\dfrac{1}{C\omega} = |\bar{Z}_{\text{tot}}|(\cos\varphi + j\sin\varphi)$ at $\varphi = -60°$:

$$R + r_0 = |\bar{Z}_{\text{tot}}|\cos(-60°) = 200 \times 0.50 = 100\,\Omega \implies r_0 = 100 - R = 100 - 80 = 20\,\Omega.$$

Equating imaginary parts with $\sin(-60°) = -\sqrt{3}/2$ and $\omega = \dfrac{1000\pi}{3}\,\text{rad}\,\text{s}^{-1}$:

$$-\frac{1}{C\omega} = |\bar{Z}_{\text{tot}}|\sin(-60°) = 200\left(-\frac{\sqrt{3}}{2}\right) = -100\sqrt{3}\,\Omega \approx -173.21\,\Omega$$

$$\implies \quad C = \frac{1}{100\sqrt{3}\,\omega} = \frac{1}{100\sqrt{3}\left(\dfrac{1000\pi}{3}\right)} = \frac{3}{10^5\pi\sqrt{3}} = \frac{\sqrt{3}}{10^5\pi}\,\text{F} \approx 5.513 \times 10^{-6}\,\text{F} \approx 5.51\,\mu\text{F}.$$

$$|\bar{Z}_{\text{tot}}| = 200\,\Omega, \qquad r_0 = 20\,\Omega, \qquad C = \frac{\sqrt{3}}{10^5\pi}\,\text{F} \approx 5.51\,\mu\text{F}$$

> ✓ **Sanity Check:**
> Pythagorean impedance check: $\sqrt{(R + r_0)^2 + X_C^2} = \sqrt{100^2 + (100\sqrt{3})^2} = \sqrt{10\,000 + 30\,000} = \sqrt{40\,000} = 200\,\Omega = |\bar{Z}_{\text{tot}}|$, and $r_0 = 20\,\Omega > 0$ confirms a passive real capacitor/RC dipole.

Exercise 7

Configuration Summary: *Series dipoles D_1, D_2 carrying $i(t) = 5\sqrt{2}\sin(100\pi t + \pi/4)$ A, with total voltage $u(t) = u_1(t) + u_2(t) = 60\sqrt{2}\sin(100\pi t + \pi/3)$ V and $u_1(t) = 80\sqrt{2}\sin(100\pi t - \pi/6)$ V.*

1) Give the RMS Fresnel vectors $\vec{U}$ and $\vec{U}_1$ (moduli and polar angles in degrees). ★☆☆☆☆

Method ***RMS Modulus Extraction and Radian-to-Degree Polar Angle Conversion***

> ***Why we do this (Way of Thinking):***
> All three given waveforms are already expressed in the common $\sin(100\pi t + \varphi)$ reference with explicit $X\sqrt{2}$ peak factors, so each RMS Fresnel modulus equals $X_m/\sqrt{2} = X$ directly.

Execution: Dividing each peak amplitude by $\sqrt{2}$ and converting initial phases via $180^\circ/\pi$:

$$U = \frac{60\sqrt{2}}{\sqrt{2}} = 60\,\text{V}, \qquad \varphi_u = +\frac{\pi}{3}\,\text{rad} = +60^\circ \qquad \Longrightarrow \quad \vec{U}\big(60\,\text{V},\ +60^\circ\big),$$

$$U_1 = \frac{80\sqrt{2}}{\sqrt{2}} = 80\,\text{V}, \qquad \varphi_{u_1} = -\frac{\pi}{6}\,\text{rad} = -30^\circ \qquad \Longrightarrow \quad \vec{U}_1\big(80\,\text{V},\ -30^\circ\big).$$

$$\boxed{\vec{U} = \big(60\,\text{V},\ +60^\circ\big), \qquad \vec{U}_1 = \big(80\,\text{V},\ -30^\circ\big)}$$

> ✓ **Sanity Check:**
> Notice right away that $U = 60\,\text{V} < U_1 = 80\,\text{V}$: in AC series circuits, the total RMS voltage U across two dipoles can be *smaller* than an individual dipole voltage U_1 whenever $u_1(t)$ and $u_2(t)$ partially cancel in phase (reactive opposition).

2) Using the scale $20\,\text{V}\,\text{cm}^{-1}$, construct to scale in the Fresnel plane the vector $\vec{U}_2$ representing $u_2(t)$.

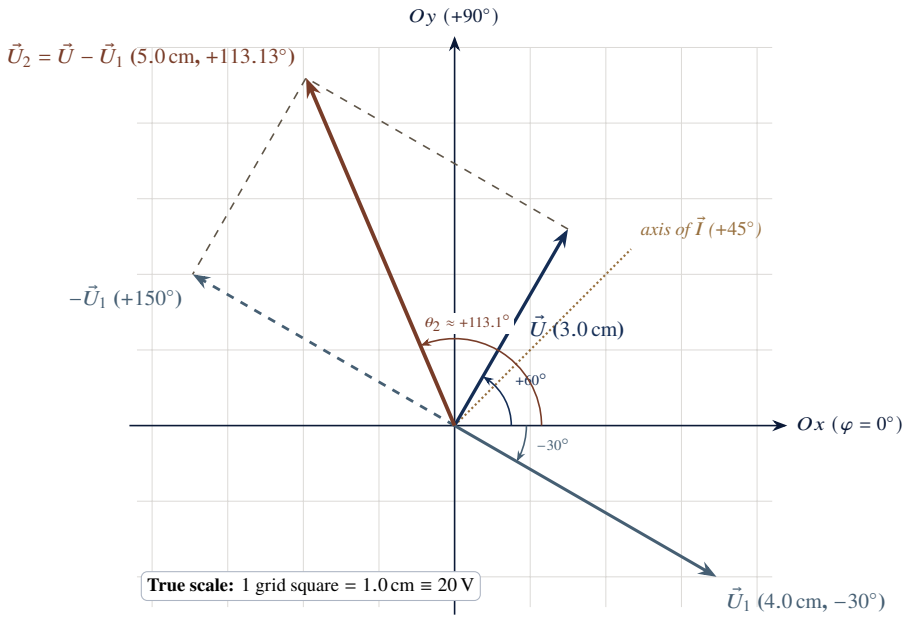
★★☆☆☆

Method ***Fresnel Vector Subtraction $\vec{U}_2 = \vec{U} - \vec{U}_1 = \vec{U} + (-\vec{U}_1)$ at True 1 cm Scale***

> ***Why we do this (Way of Thinking):***
> Time-domain KVL $u(t) = u_1(t) + u_2(t)$ maps isomorphically onto the Fresnel vector equation $\vec{U} = \vec{U}_1 + \vec{U}_2$, so $\vec{U}_2 = \vec{U} - \vec{U}_1$. Graphically, we draw $\vec{U}$ ($\ell_U = 60/20 = 3.0\,\text{cm}$ at $+60^\circ$) and $\vec{U}_1$ ($\ell_{U_1} = 80/20 = 4.0\,\text{cm}$ at -30°), reverse $\vec{U}_1$ by 180° to obtain $-\vec{U}_1$ ($4.0\,\text{cm}$ at $-30^\circ + 180^\circ = +150^\circ$), and complete the parallelogram diagonal $\vec{U}_2 = \vec{U} + (-\vec{U}_1)$.

Execution: At scale $20\,\text{V}\,\text{cm}^{-1}$:

$$\ell_U = \frac{60\,\text{V}}{20\,\text{V}\,\text{cm}^{-1}} = 3.0\,\text{cm}\ (\text{at}\ +60^\circ), \qquad \ell_{U_1} = \frac{80\,\text{V}}{20\,\text{V}\,\text{cm}^{-1}} = 4.0\,\text{cm}\ (\text{at}\ -30^\circ \implies -\vec{U}_1\ \text{at}\ +150^\circ).$$



Measuring the resultant diagonal on the 1 cm grid gives $\ell_{U_2} = 5.0\,\text{cm}$ ($U_2 = 5.0 \times 20 = 100\,\text{V}$) in Quadrant II at $\theta_2 \approx +113^\circ$.

$$\boxed{\ell_{U_2} = 5.0\,\text{cm} \implies U_2 = 100\,\text{V in Quadrant II at } \theta_2 \approx +113.1^\circ}$$

> ✓ **Sanity Check:**
> Observe the geometric orthogonality on the diagram: $\vec{U}$ sits at $+60°$ while $-\vec{U}_1$ sits at $+150°$, separated by $150° - 60° = 90°$. Hence $\vec{U}$ (3.0 cm) and $-\vec{U}_1$ (4.0 cm) form the legs of an exact 3–4–5 right-angled rectangle whose diagonal $\vec{U}_2$ has exact length $\sqrt{3^2 + 4^2} = 5.0$ cm.

3) Calculate the rectangular projections (U_{2x}, U_{2y}), the RMS voltage U_2, and the initial phase $\theta_2 \in (-180°, +180°]$ of $u_2(t)$. Write the time-domain expression $u_2(t)$. ★★★☆☆

Method ***Cartesian Fresnel Projection and Two-Argument Quadrant-Aware*** $\operatorname{atan2}(U_{2y}, U_{2x})$

> ***Why we do this (Way of Thinking):***
> Projecting $\vec{U}_2 = \vec{U} - \vec{U}_1$ onto (Ox, Oy) turns vector subtraction into exact component algebra. Whenever the horizontal projection U_{2x} turns out negative, the principal calculator value $\arctan(U_{2y}/U_{2x}) \in (-90°, +90°)$ lands in Quadrant I or IV and must be shifted by $+180°$ (for $U_{2y} \geq 0$, Quadrant II) or $-180°$ (for $U_{2y} < 0$, Quadrant III).

Execution:

Step 1 — Rectangular projections (U_{2x}, U_{2y}):

$$U_{2x} = U\cos(60°) - U_1\cos(-30°) = 60\left(\frac{1}{2}\right) - 80\left(\frac{\sqrt{3}}{2}\right) = (30 - 40\sqrt{3})\ \text{V} \approx -39.282\ \text{V} < 0,$$

$$U_{2y} = U\sin(60°) - U_1\sin(-30°) = 60\left(\frac{\sqrt{3}}{2}\right) - 80\left(-\frac{1}{2}\right) = (30\sqrt{3} + 40)\ \text{V} \approx +91.962\ \text{V} > 0.$$

Step 2 — Exact RMS modulus U_2: Expanding $U_{2x}^2 + U_{2y}^2$ algebraically without rounding errors:

$$\begin{aligned} U_2^2 &= (30 - 40\sqrt{3})^2 + (30\sqrt{3} + 40)^2 \\ &= \left(900 - 2400\sqrt{3} + 4800\right) + \left(2700 + 2400\sqrt{3} + 1600\right) = 5700 + 4300 = 10\,000\ \text{V}^2 \\ &\Longrightarrow \quad U_2 = \sqrt{10\,000} = 100\ \text{V}. \end{aligned}$$

Step 3 — Quadrant-corrected phase θ_2 and time-domain waveform $u_2(t)$: Because $U_{2x} < 0$ and $U_{2y} > 0$, $\vec{U}_2$ lies strictly in **Quadrant II**, so $\theta_2 = \arctan(U_{2y}/U_{2x}) + 180°$:

$$\theta_2 = \arctan\left(\frac{30\sqrt{3} + 40}{30 - 40\sqrt{3}}\right) + 180° \approx -66.87° + 180° = +113.13° \quad (\approx +1.974\ \text{rad}).$$

With peak amplitude $U_{2m} = U_2\sqrt{2} = 100\sqrt{2}\ \text{V}$ and $\omega = 100\pi\ \text{rad s}^{-1}$:

$$u_2(t) = 100\sqrt{2}\sin(100\pi t + 1.974)\ \text{V}.$$

$$U_{2x} \approx -39.28\ \text{V},\ U_{2y} \approx +91.96\ \text{V},\ U_2 = 100\ \text{V},\ \theta_2 \approx +113.13°\ (+1.974\ \text{rad}),$$
$$u_2(t) = 100\sqrt{2}\sin(100\pi t + 1.974)\ \text{V}$$

> ► **Common Pitfall:**
> Writing $\theta_2 = \arctan(-2.341) = -66.87°$ without checking the signs of (U_{2x}, U_{2y}) places $\vec{U}_2$ in Quadrant IV ($-\vec{U}_2$) and violates KVL completely ($u_1 + u_2$ would yield 124.2 V instead of 60 V). Always add $+180°$ (π rad) whenever $U_{2x} < 0$ and $U_{2y} \geq 0$.

4) Verify U_2 using the Fresnel law of cosines. ★☆☆☆☆

Method ***Fresnel Law of Cosines for Vector Difference $\vec{U}_2 = \vec{U} - \vec{U}_1$***

> ***Why we do this (Way of Thinking):***
> For any vector difference $\vec{U}_2 = \vec{U} - \vec{U}_1$, the squared modulus satisfies $U_2^2 = U^2 + U_1^2 - 2\,U\,U_1\cos(\varphi_u - \varphi_{u_1})$ (with a *minus* sign before $2UU_1\cos\Delta\theta$, unlike vector addition $\vec{U} = \vec{U}_1 + \vec{U}_2$ which carries $+2U_1U_2\cos(\varphi_{u_1} - \theta_2)$).

Execution: With $\Delta\theta = \varphi_u - \varphi_{u_1} = (+60°) - (-30°) = +90°$ ($\cos 90° = 0$):

$$\begin{aligned} U_2 &= \sqrt{U^2 + U_1^2 - 2\,U\,U_1\cos(\varphi_u - \varphi_{u_1})} \\ &= \sqrt{60^2 + 80^2 - 2(60)(80)\cos 90°} = \sqrt{3600 + 6400 - 0} = \sqrt{10\,000} = 100\,\text{V}. \end{aligned}$$

$$\boxed{U_2 = \sqrt{60^2 + 80^2 - 2(60)(80)\cos 90°} = 100\,\text{V}}$$

> ✓ **Sanity Check:**
> Checking the forward addition law $\vec{U} = \vec{U}_1 + \vec{U}_2$ as well: $\theta_2 - \varphi_{u_1} = 113.13° - (-30°) = 143.13°$ (where $\cos 143.13° = -4/5 = -0.80$), so $\sqrt{U_1^2 + U_2^2 + 2U_1U_2\cos 143.13°} = \sqrt{6400 + 10\,000 + 2(80)(100)(-0.80)} = \sqrt{16\,400 - 12\,800} = \sqrt{3600} = 60\,\text{V} = U$.

5) Compute the relative phase shifts φ_{u/u_1}, $\varphi_{u_1/i}$, $\varphi_{u_2/i}$, and $\varphi_{u/i}$, and check the additive phase relation linking φ_{u/u_1}, $\varphi_{u_1/i}$, and $\varphi_{u/i}$. ★★☆☆☆

Method ***Pairwise Phase Differences $\varphi_{a/b} = \varphi_a - \varphi_b$ and Telescoping Phase Identity***

> ***Why we do this (Way of Thinking):***
> In a series loop sharing current $i(t)$ ($\varphi_i = +\pi/4 = +45°$), each dipole impedance angle $\varphi_{u_k/i} = \varphi_{u_k} - \varphi_i$ must lie in $[-90°, +90°]$ for passive elements, providing an immediate physical passivity test for D_1, D_2, and the combined dipole.

Execution: Using $\varphi_i = +45°$, $\varphi_u = +60°$, $\varphi_{u_1} = -30°$, and $\theta_2 = +113.13°$:

$$\begin{aligned} \varphi_{u/u_1} &= \varphi_u - \varphi_{u_1} = 60° - (-30°) = +90° \quad \left(+\frac{\pi}{2}\ \text{rad}\right), \\ \varphi_{u_1/i} &= \varphi_{u_1} - \varphi_i = -30° - 45° = -75° \quad \left(-\frac{5\pi}{12}\ \text{rad},\ \text{capacitive } D_1\right), \\ \varphi_{u_2/i} &= \theta_2 - \varphi_i = 113.13° - 45° = +68.13° \quad (+1.189\,\text{rad},\ \text{inductive } D_2), \\ \varphi_{u/i} &= \varphi_u - \varphi_i = 60° - 45° = +15° \quad \left(+\frac{\pi}{12}\ \text{rad},\ \text{net inductive combination}\right). \end{aligned}$$

Checking the additive telescoping phase relation:

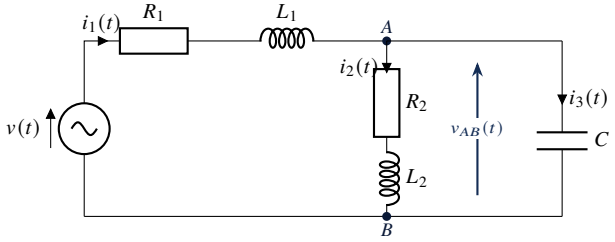
$$\varphi_{u/u_1} + \varphi_{u_1/i} = (\varphi_u - \varphi_{u_1}) + (\varphi_{u_1} - \varphi_i) = (+90°) + (-75°) = +15° = \varphi_{u/i}.$$

$$\boxed{\varphi_{u/u_1} = +90°,\ \ \varphi_{u_1/i} = -75°,\ \ \varphi_{u_2/i} \approx +68.13°,\ \ \varphi_{u/i} = +15°,\ \ \varphi_{u/u_1} + \varphi_{u_1/i} = \varphi_{u/i}}$$

> ✓ **Sanity Check:**
> Every dipole impedance angle satisfies $|\varphi_{u_k/i}| \le 90°$ ($|-75°| \le 90°$, $|+68.13°| \le 90°$, $|+15°| \le 90°$), confirming that D_1, D_2, and their series association are all strictly passive ($\cos\varphi \ge 0$).

Exercise 8

Configuration Summary: *Mixed series–parallel AC network fed by* $\bar{V} = 150\angle 0^\circ\,\mathrm{V}$ *at* $\omega = 1000\,\mathrm{rad\,s^{-1}}$*, with Branch 1 (*$R_1 = 7\,\Omega$*,* $L_1 = 24\,\mathrm{mH}$*) in series with parallel block* (A, B) *formed by Branch 2 (*$R_2 = 25\,\Omega$*,* $L_2 = 25\,\mathrm{mH}$*) and Branch 3 (*C*).*



1) Calculate the complex impedances $\bar{Z}_1$ and $\bar{Z}_2$, then determine the complex admittance $\bar{Y}_2 = G_2 + jB_2$ of Branch 2 (R_2, L_2). Compare G_2 with $1/R_2$. ★★☆☆☆

Method ***Complex Series Impedance $\bar{Z} = R + jL\omega$ and Complex Conjugate Rationalization $\bar{Y} = \bar{Z}^*/|\bar{Z}|^2$***

> ***Why we do this (Way of Thinking):***
> Because Branch 2 (R_2, L_2) sits in parallel with capacitor C between nodes A and B, converting $\bar{Z}_2$ into its complex admittance $\bar{Y}_2 = 1/\bar{Z}_2 = G_2 + jB_2$ via conjugate multiplication $(R_2 - jX_2)/(R_2^2 + X_2^2)$ makes the parallel combination across (A, B) simple complex addition.

Execution:

Step 1 — Complex series impedances $\bar{Z}_1$ and $\bar{Z}_2$ at $\omega = 1000\,\mathrm{rad\,s^{-1}}$:

$$X_{L1} = L_1\omega = (24\times 10^{-3}\,\mathrm{H})(1000\,\mathrm{rad\,s^{-1}}) = 24\,\Omega \implies \bar{Z}_1 = R_1 + jX_{L1} = (7 + j\,24)\,\Omega,$$
$$X_{L2} = L_2\omega = (25\times 10^{-3}\,\mathrm{H})(1000\,\mathrm{rad\,s^{-1}}) = 25\,\Omega \implies \bar{Z}_2 = R_2 + jX_{L2} = (25 + j\,25)\,\Omega.$$

Step 2 — Complex admittance $\bar{Y}_2 = G_2 + jB_2$ and comparison of G_2 with $1/R_2$:

Sub **Multiply numerator and denominator of $1/(R_2 + jX_{L2})$ by the complex conjugate $(R_2 - jX_{L2})$:**

$$\bar{Y}_2 = \frac{1}{\bar{Z}_2} = \frac{1}{25 + j\,25} = \frac{25 - j\,25}{25^2 + 25^2} = \frac{25 - j\,25}{625 + 625} = \frac{25 - j\,25}{1250} = (0.02 - j\,0.02)\ \mathrm{S}.$$

Hence $G_2 = 0.02\,\mathrm{S}$ and $B_2 = -0.02\,\mathrm{S}$. Comparing G_2 against $1/R_2$:

$$G_2 = \frac{R_2}{R_2^2 + X_{L2}^2} = 0.02\,\mathrm{S} \neq \frac{1}{R_2} = \frac{1}{25} = 0.04\,\mathrm{S} \quad \big(G_2 \text{ is half of } 1/R_2 \text{ because } X_{L2} = R_2 \neq 0\big).$$

$$\bar{Z}_1 = (7 + j\,24)\,\Omega, \qquad \bar{Z}_2 = (25 + j\,25)\,\Omega, \qquad \bar{Y}_2 = (0.02 - j\,0.02)\ \mathrm{S}\ \big(G_2 = 0.02\,\mathrm{S} \neq 1/R_2\big)$$

> ► **Common Pitfall:**
> Writing $\bar{Y}_2 = \frac{1}{R_2} + \frac{1}{jL_2\omega} = 0.04 - j\,0.04\,\mathrm{S}$ treats a *series* RL branch as if R_2 and L_2 were in *parallel*. For a series branch $\bar{Z} = R + jX$, $G = R/(R^2 + X^2)$ equals $1/R$ if and only if $X = 0$.

2) For $C = 80\,\mu\mathrm{F}$, calculate the parallel admittance $\bar{Y}_{AB} = \bar{Y}_2 + \bar{Y}_3$, the parallel impedance $\bar{Z}_{AB} = 1/\bar{Y}_{AB}$, and the total input impedance $\bar{Z}_{\mathrm{eq}} = \bar{Z}_1 + \bar{Z}_{AB}$ in both rectangular and polar forms. Deduce the quality factor Q of the complete dipole seen by the source. ★★★☆☆

Method ***Admittance Addition Across Parallel Nodes Followed by Series Impedance Addition***

> ***Why we do this (Way of Thinking):***
> Alternating cleanly between the admittance domain for parallel blocks ($\bar{Y}_{AB} = \bar{Y}_2 + \bar{Y}_3$) and the impedance domain for series blocks ($\bar{Z}_{\mathrm{eq}} = \bar{Z}_1 + 1/\bar{Y}_{AB}$) avoids error-prone complex product-over-sum fractions. Once $\bar{Z}_{\mathrm{eq}} = R_{\mathrm{eq}} + jX_{\mathrm{eq}}$ is obtained, the dipole quality factor follows as $Q = |X_{\mathrm{eq}}|/R_{\mathrm{eq}} = |\tan\varphi|$.

Execution:

Step 1 — Capacitive admittance $\bar{Y}_3$, parallel admittance $\bar{Y}_{AB}$, and $\bar{Z}_{AB}$: For $C = 80\,\mu\text{F} = 8.0 \times 10^{-5}\,\text{F}$ at $\omega = 1000\,\text{rad}\,\text{s}^{-1}$:

$$\bar{Y}_3 = j\,C\omega = j(8.0 \times 10^{-5}\,\text{F})(1000\,\text{rad}\,\text{s}^{-1}) = +j\,0.08\ \text{S},$$
$$\bar{Y}_{AB} = \bar{Y}_2 + \bar{Y}_3 = (0.02 - j\,0.02) + j\,0.08 = (0.02 + j\,0.06)\ \text{S}.$$

Inverting $\bar{Y}_{AB}$ by conjugate multiplication:

$$\bar{Z}_{AB} = \frac{1}{\bar{Y}_{AB}} = \frac{1}{0.02 + j\,0.06} = \frac{0.02 - j\,0.06}{0.02^2 + 0.06^2} = \frac{0.02 - j\,0.06}{0.0004 + 0.0036} = \frac{0.02 - j\,0.06}{0.0040} = (5 - j\,15)\ \Omega.$$

Step 2 — Total input impedance $\bar{Z}_{\text{eq}}$ (rectangular & polar) and quality factor Q: Adding $\bar{Z}_1 = (7 + j\,24)\ \Omega$ in series with $\bar{Z}_{AB} = (5 - j\,15)\ \Omega$:

$$\bar{Z}_{\text{eq}} = \bar{Z}_1 + \bar{Z}_{AB} = (7 + 5) + j(24 - 15) = (12 + j\,9)\ \Omega.$$

Converting $\bar{Z}_{\text{eq}}$ to polar form ($R_{\text{eq}} = 12 > 0$, $X_{\text{eq}} = +9 > 0$, Quadrant I):

$$|\bar{Z}_{\text{eq}}| = \sqrt{12^2 + 9^2} = \sqrt{144 + 81} = \sqrt{225} = 15\,\Omega,$$
$$\varphi = \arctan\left(\frac{9}{12}\right) = \arctan(0.75) \approx +36.87^\circ \quad (\approx +0.6435\,\text{rad}),$$
$$\bar{Z}_{\text{eq}} = 15\angle{+36.87^\circ}\ \Omega.$$

The quality factor (reactive-to-resistive storage ratio) of the complete equivalent dipole is:

$$Q = \frac{|X_{\text{eq}}|}{R_{\text{eq}}} = |\tan\varphi| = \frac{9}{12} = 0.75.$$

$\bar{Y}_{AB} = (0.02 + j\,0.06)\ \text{S},\ \ \bar{Z}_{AB} = (5 - j\,15)\ \Omega,\ \ \bar{Z}_{\text{eq}} = (12 + j\,9)\ \Omega = 15\angle{+36.87^\circ}\ \Omega,\ \ Q = 0.75$

✓ **Sanity Check:**
Inside parallel block (A, B), the capacitive susceptance $+0.08\,\text{S}$ dominates the inductive susceptance $-0.02\,\text{S}$, making $\bar{Z}_{AB} = (5 - j\,15)\ \Omega$ capacitive, whereas upstream series reactance $+j\,24\,\Omega$ outweighs $-j\,15\,\Omega$ to leave the overall input impedance $\bar{Z}_{\text{eq}} = (12 + j\,9)\ \Omega$ net inductive.

3) Still for $C = 80\,\mu\text{F}$, determine the complex phasors $\bar{I}_1$, $\bar{V}_{AB}$, $\bar{I}_2$, and $\bar{I}_3$, and write the instantaneous source current $i_1(t)$. ★★★☆☆

Method ***Complex Ohm's Law Cascade*** *($\bar{I}_1 = \bar{V}/\bar{Z}_{\text{eq}} \longrightarrow \bar{V}_{AB} = \bar{Z}_{AB}\bar{I}_1 \longrightarrow \bar{I}_k = \bar{Y}_k\bar{V}_{AB}$)*

Why we do this (Way of Thinking):
Once $\bar{Z}_{\text{eq}}$, $\bar{Z}_{AB}$, $\bar{Y}_2$, and $\bar{Y}_3$ are known in exact rectangular form, every branch phasor follows by a single complex multiplication without solving simultaneous complex loop equations.

Execution:

Step 1 — Source current phasor $\bar{I}_1$ and instantaneous current $i_1(t)$: With $\bar{V} = 150\angle{0^\circ} = (150 + j\,0)\ \text{V}$ and $\bar{Z}_{\text{eq}} = (12 + j\,9)\ \Omega = 15\angle{+36.87^\circ}\ \Omega$:

$$\bar{I}_1 = \frac{\bar{V}}{\bar{Z}_{\text{eq}}} = \frac{150}{12 + j\,9} = \frac{150(12 - j\,9)}{12^2 + 9^2} = \frac{150(12 - j\,9)}{225} = \frac{2}{3}(12 - j\,9) = (8 - j\,6)\ \text{A} = 10\angle{-36.87^\circ}\ \text{A}.$$

Since $I_1 = 10\,\text{A}$ (RMS) $\Longrightarrow$ $I_{1m} = 10\sqrt{2}\,\text{A}$ and $\varphi_{i_1} = -\arctan(3/4) \approx -0.6435\,\text{rad}$ (-36.87°):

$$i_1(t) = 10\sqrt{2}\sin(1000\,t - 0.6435)\ \text{A}.$$

Step 2 — Parallel node voltage $\bar{V}_{AB}$ and branch currents $\bar{I}_2$, $\bar{I}_3$: Using $j^2 = -1$ at each complex product:

$$\begin{aligned}
\bar{V}_{AB} &= \bar{Z}_{AB}\,\bar{I}_1 = (5 - j\,15)(8 - j\,6) = (40 - 90) + j(-30 - 120) = (-50 - j\,150)\ \text{V} \\
&= 50\sqrt{10}\angle -108.43^\circ\ \text{V} \approx 158.11\angle -108.43^\circ\ \text{V}, \\
\bar{I}_2 &= \bar{Y}_2\,\bar{V}_{AB} = (0.02 - j\,0.02)(-50 - j\,150) = (-1.0 - 3.0) + j(-3.0 + 1.0) = (-4 - j\,2)\ \text{A} \\
&= 2\sqrt{5}\angle -153.43^\circ\ \text{A} \approx 4.472\angle -153.43^\circ\ \text{A}, \\
\bar{I}_3 &= \bar{Y}_3\,\bar{V}_{AB} = (j\,0.08)(-50 - j\,150) = 12 - j\,4\ \text{A} \\
&= 4\sqrt{10}\angle -18.43^\circ\ \text{A} \approx 12.649\angle -18.43^\circ\ \text{A}.
\end{aligned}$$

$$\bar{I}_1 = (8 - j\,6)\ \text{A},\quad \bar{V}_{AB} = (-50 - j\,150)\ \text{V},\quad \bar{I}_2 = (-4 - j\,2)\ \text{A},\quad \bar{I}_3 = (12 - j\,4)\ \text{A}$$

$$i_1(t) = 10\sqrt{2}\sin(1000\,t - 0.6435)\ \text{A}$$

> **► Common Pitfall:**
> Note the Quadrant-III angles of $\bar{V}_{AB} = -50 - j\,150$ and $\bar{I}_2 = -4 - j\,2$: both real and imaginary parts are negative, so one must subtract 180° from $\arctan(+3) = +71.57°$ and $\arctan(+0.5) = +26.57°$, giving $-108.43°$ and $-153.43°$.

4) Verify complex KCL at node A and complex KVL around the outer mesh, and compare the scalar sum of RMS branch currents $I_2 + I_3$ with I_1. ★★☆☆☆

Method ***Complex Phasor Kirchhoff Verification vs. Scalar RMS Triangle Inequality***

> ***Why we do this (Way of Thinking):***
> In AC steady state, KCL and KVL hold algebraically for *complex phasors* ($\bar{I}_1 = \bar{I}_2 + \bar{I}_3$ and $\bar{V} = \bar{Z}_1\bar{I}_1 + \bar{V}_{AB}$), never for scalar RMS meter readings whenever the branch signals are out of phase.

Execution:

- **Complex KCL at node A:**

$$\bar{I}_2 + \bar{I}_3 = (-4 - j\,2) + (12 - j\,4) = (8 - j\,6)\ \text{A} = \bar{I}_1. \quad \checkmark$$

- **Complex KVL around the outer mesh:** First compute $\bar{U}_1 = \bar{Z}_1\,\bar{I}_1$:

$$\bar{Z}_1\,\bar{I}_1 = (7 + j\,24)(8 - j\,6) = (56 + 144) + j(-42 + 192) = (200 + j\,150)\ \text{V},$$

  and add $\bar{V}_{AB} = (-50 - j\,150)$ V:

$$\bar{Z}_1\,\bar{I}_1 + \bar{V}_{AB} = (200 - 50) + j(150 - 150) = (150 + j\,0)\ \text{V} = \bar{V}. \quad \checkmark$$

- **Scalar RMS comparison:** With $I_2 = |\bar{I}_2| = 2\sqrt{5} \approx 4.472$ A and $I_3 = |\bar{I}_3| = 4\sqrt{10} \approx 12.649$ A:

$$I_2 + I_3 = 2\sqrt{5} + 4\sqrt{10} \approx 4.472 + 12.649 = 17.12\ \text{A} > I_1 = 10.00\ \text{A} \quad (I_2 + I_3 \neq I_1).$$

$$\bar{I}_2 + \bar{I}_3 = \bar{I}_1,\ \ \bar{Z}_1\bar{I}_1 + \bar{V}_{AB} = \bar{V},\ \ \text{whereas } I_2 + I_3 \approx 17.12\ \text{A} > I_1 = 10.00\ \text{A}$$

> **✓ Sanity Check:**
> Even the capacitor RMS current alone ($I_3 \approx 12.65$ A) exceeds the total line current ($I_1 = 10.00$ A), and the node RMS voltage ($V_{AB} \approx 158.1$ V) exceeds the supply voltage ($V = 150$ V): reactive energy oscillates locally between L_2 (L_1) and C with a 135° phase opposition between $\bar{I}_2$ and $\bar{I}_3$.

5) Keeping $(R_1, L_1, R_2, L_2, \omega)$ fixed, determine all values of the capacitance C_0 for which the source current $i_1(t)$ is in phase with $v(t)$, and give the corresponding purely resistive input impedance $Z_{\text{eq}}(C_0)$ in each case. ★★★★☆

Method ***Phase-Resonance Condition*** $\text{Im}\big(\bar{Z}_{\text{eq}}(C_0)\big) = 0$ ***via Net Parallel Susceptance*** B

> ***Why we do this (Way of Thinking):***
> Source current $i_1(t)$ is in phase with $v(t)$ ($\varphi = 0°$) if and only if the total input impedance $\bar{Z}_{\text{eq}}(C_0)$ is purely real ($\text{Im}(\bar{Z}_{\text{eq}}) = 0$). Parameterizing the parallel block by its net susceptance $B = C_0\omega + B_2 = C_0\omega - 0.02$ turns $\text{Im}(\bar{Z}_1 + 1/(G_2 + jB)) = 0$ into a clean quadratic equation in B, revealing *two* distinct resonant capacitances whenever X_{L1} is small enough for the parallel tank to reach $-X_{L1}$.

Execution:

Step 1 — Express $\bar{Z}_{\text{eq}}(C_0)$ in terms of the net parallel susceptance B:

Sub **Set $B = C_0\omega - 0.02$ (in S), so $\bar{Y}_{AB} = G_2 + jB = 0.02 + jB$ with $G_2 = 0.02\,\text{S}$ and $G_2^2 = 0.0004\,\text{S}^2$.**

Rationalizing $\bar{Z}_{AB} = 1/(0.02 + jB)$ and adding $\bar{Z}_1 = 7 + j\,24$:

$$\bar{Z}_{\text{eq}}(C_0) = 7 + j\,24 + \frac{0.02 - jB}{0.0004 + B^2} = \left(7 + \frac{0.02}{0.0004 + B^2}\right) + j\left(24 - \frac{B}{0.0004 + B^2}\right).$$

Step 2 — Impose $\text{Im}\big(\bar{Z}_{\text{eq}}(C_0)\big) = 0$ and solve the quadratic in B:

$$24 - \frac{B}{0.0004 + B^2} = 0 \iff 24(0.0004 + B^2) = B$$
$$\iff 24\,B^2 - B + 0.0096 = 0.$$

Computing the discriminant Δ:

$$\Delta = (-1)^2 - 4(24)(0.0096) = 1 - 0.9216 = 0.0784 = (0.28)^2 > 0, \qquad \sqrt{\Delta} = 0.28.$$

Hence two real positive susceptance roots exist:

$$B^{(1)} = \frac{1 - 0.28}{48} = \frac{0.72}{48} = 0.015\,\text{S}, \qquad B^{(2)} = \frac{1 + 0.28}{48} = \frac{1.28}{48} = \frac{2}{75}\,\text{S} \approx 0.026\,67\,\text{S}.$$

Step 3 — Deduce $C_0^{(1)}, C_0^{(2)}$ and the purely resistive impedances $Z_{\text{eq}}(C_0)$: From $C_0 = \dfrac{B + 0.02}{\omega}$ at $\omega = 1000\,\text{rad}\,\text{s}^{-1}$, and noticing from the resonance equation that $0.0004 + B^2 = \dfrac{B}{24}$ so $Z_{\text{eq}} = 7 + \dfrac{0.02}{B/24} = 7 + \dfrac{0.48}{B}$:

- **First solution:**

$$C_0^{(1)} = \frac{0.015 + 0.02}{1000} = 3.5 \times 10^{-5}\,\text{F} = 35.0\,\mu\text{F}, \qquad Z_{\text{eq}}\big(C_0^{(1)}\big) = 7 + \frac{0.48}{0.015} = 7 + 32 = 39.0\,\Omega.$$

- **Second solution:**

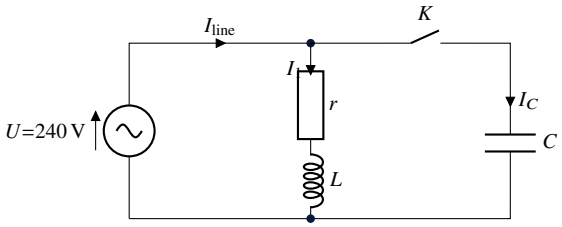
$$C_0^{(2)} = \frac{2/75 + 1/50}{1000} = \frac{7/150}{1000} = \frac{7}{150\,000}\,\text{F} = \frac{140}{3}\,\mu\text{F} \approx 46.67\,\mu\text{F},$$
$$Z_{\text{eq}}\big(C_0^{(2)}\big) = 7 + \frac{0.48}{2/75} = 7 + 18 = 25.0\,\Omega.$$

$$C_0^{(1)} = 35.0\,\mu\text{F} \implies Z_{\text{eq}} = 39.0\,\Omega; \qquad C_0^{(2)} = \frac{140}{3}\,\mu\text{F} \approx 46.67\,\mu\text{F} \implies Z_{\text{eq}} = 25.0\,\Omega$$

> ► **Common Pitfall:**
> Setting $\text{Im}(\bar{Y}_{AB}) = 0$ ($B = 0 \implies C_0 = 20\,\mu\text{F}$) resonates only the *parallel block* (A, B), leaving $\bar{Z}_{\text{eq}} = (7 + 50) + j\,24 = (57 + j\,24)\,\Omega$ still inductive because of the series coil L_1 upstream. One must set $\text{Im}(\bar{Z}_1 + \bar{Z}_{AB}) = 0$ so that $\text{Im}(\bar{Z}_{AB}) = -24\,\Omega$ cancels $+jL_1\omega$.

Exercise 9

Configuration Summary: *Industrial coil (r, L) connected across an AC bus ($U = 240\,\text{V}$, $f = 50\,\text{Hz}$); with switch K open, $I_1 = 6.0\,\text{A}$ and $P = 864\,\text{W}$. Closing switch K adds an ideal parallel capacitor C for power-factor compensation.*



1) With switch K open: (a) Calculate S_1, $\cos\varphi_1$, signed φ_1, and Q_{r1} (with units), and explain why $\cos\varphi_1$ alone does not fix the sign of φ_1. (b) Determine $|\bar{Z}_{\text{coil}}|$, r, X_L, and L. (c) Calculate $U_r = rI_1$ and $U_L = X_L I_1$, and verify the Fresnel voltage relation. ★★☆☆☆

Method ***AC Power Triangle ($S^2 = P^2 + Q_r^2$), Boucherot Relations, and Fresnel Voltage Triangle***

> ***Why we do this (Way of Thinking):***
> An ammeter, voltmeter, and wattmeter provide (U, I_1, P), from which apparent power $S_1 = UI_1$ and power factor $\cos\varphi_1 = P/S_1$ follow directly. Because $\cos\varphi$ is an *even* function, $\cos\varphi_1$ fixes only $|\varphi_1|$; the inductive physics of the coil ($X_L = L\omega > 0$) selects $\varphi_1 > 0$ ($\sin\varphi_1 > 0$, $Q_{r1} > 0$). Expressing active power as $P = rI_1^2$ and reactive power as $Q_{r1} = X_L I_1^2$ then isolates (r, X_L, L) directly.

Execution:

(a) Apparent power S_1, power factor $\cos\varphi_1$, signed angle φ_1, and reactive power Q_{r1}:

$$S_1 = U\,I_1 = (240\,\text{V})(6.0\,\text{A}) = 1440\,\text{VA} \quad \text{(volt-ampere)},$$
$$\cos\varphi_1 = \frac{P}{S_1} = \frac{864\,\text{W}}{1440\,\text{VA}} = 0.60 \quad \text{(dimensionless)}.$$

Sign ambiguity of $\cos\varphi_1$: Because $\cos(-\varphi_1) = \cos(+\varphi_1)$, knowing $\cos\varphi_1 = 0.60$ alone yields two symmetric angles $\varphi_1 = \pm\arccos(0.60) \approx \pm 53.13^\circ$ ($\pm 0.9273\,\text{rad}$): $+53.13^\circ$ for an inductive load and -53.13° for a capacitive load. Because the industrial coil is **inductive** (u leads i_1), $\varphi_1 > 0$:

$$\varphi_1 = +\arccos(0.60) \approx +53.13^\circ \quad (+0.9273\,\text{rad}), \qquad \sin\varphi_1 = +\sqrt{1 - 0.60^2} = +\sqrt{0.64} = +0.80,$$
$$Q_{r1} = S_1 \sin\varphi_1 = 1440 \times 0.80 = +1152\,\text{var} \quad \text{(volt-ampere reactive)}.$$

(b) Coil impedance modulus $|\bar{Z}_{\text{coil}}|$, internal resistance r, reactance X_L, and inductance L: With $\omega = 2\pi f = 2\pi(50) = 100\pi\,\text{rad}\,\text{s}^{-1}$:

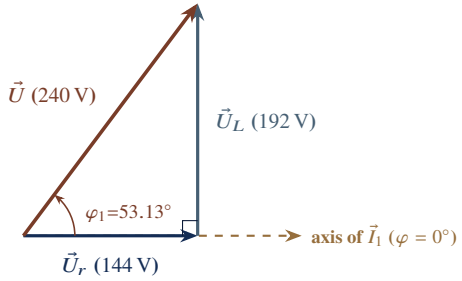
$$|\bar{Z}_{\text{coil}}| = \frac{U}{I_1} = \frac{240\,\text{V}}{6.0\,\text{A}} = 40.0\,\Omega,$$
$$r = \frac{P}{I_1^2} = \frac{864}{6.0^2} = \frac{864}{36} = 24.0\,\Omega \quad (\text{also } |\bar{Z}_{\text{coil}}|\cos\varphi_1 = 40.0 \times 0.60 = 24.0\,\Omega),$$
$$X_L = \sqrt{|\bar{Z}_{\text{coil}}|^2 - r^2} = \sqrt{40.0^2 - 24.0^2} = \sqrt{1600 - 576} = \sqrt{1024} = 32.0\,\Omega \quad (\text{also } Q_{r1}/I_1^2),$$
$$L = \frac{X_L}{\omega} = \frac{32.0}{100\pi} = \frac{8}{25\pi}\,\text{H} \approx 0.101\,86\,\text{H} \approx 101.9\,\text{mH}.$$

(c) Internal RMS voltages U_r, U_L and right-angled Fresnel voltage verification:

$$U_r = r\,I_1 = (24.0\,\Omega)(6.0\,\text{A}) = 144\,\text{V}, \qquad U_L = X_L\,I_1 = (32.0\,\Omega)(6.0\,\text{A}) = 192\,\text{V}.$$

Because $\vec{U}_r$ (in phase with $\vec{I}_1$) and $\vec{U}_L$ (leading $\vec{I}_1$ by $+90^\circ$) are mutually orthogonal, $\vec{U} = \vec{U}_r + \vec{U}_L$ obeys the Pythagorean Fresnel relation:

$$\sqrt{U_r^2 + U_L^2} = \sqrt{144^2 + 192^2} = \sqrt{20\,736 + 36\,864} = \sqrt{57\,600} = 240\,\text{V} = U. \quad \checkmark$$



$$S_1 = 1440\,\text{VA},\quad \cos\varphi_1 = 0.60,\quad \varphi_1 \approx +53.13°,\quad Q_{r1} = +1152\,\text{var}$$

$$|\bar{Z}_{\text{coil}}| = 40\,\Omega,\quad r = 24\,\Omega,\quad X_L = 32\,\Omega,\quad L \approx 101.9\,\text{mH},\quad U_r = 144\,\text{V},\quad U_L = 192\,\text{V}$$

► **Common Pitfall:**
Notice again that $U_r + U_L = 144 + 192 = 336\,\text{V} \neq 240\,\text{V}$: RMS voltages across resistive and reactive elements in series combine *in quadrature* ($U = \sqrt{U_r^2 + U_L^2}$), never by scalar addition.

2) Switch K is now closed, placing an ideal capacitor C in parallel across (U, f): (a) Express P_C and $Q_{r,C}$ in terms of (U, C, ω) and state how closing K affects total active power. (b) Determine C required for $\cos\varphi' = 1$. (c) Compute the new RMS supply current I'_{line} and the percentage reduction in line current.

★★★☆☆

Method ***Boucherot's Power Conservation Theorem*** *($P_{\text{tot}} = \sum P_k$, $Q_{r,\text{tot}} = \sum Q_{r,k}$)*

Why we do this (Way of Thinking):
Boucherot's theorem eliminates all complex-admittance algebra in parallel power-factor correction: connecting an ideal capacitor C in parallel across the fixed bus $U = 240\,\text{V}$ leaves the coil's terminal voltage U, coil current I_1, coil active power P, and coil reactive power Q_{r1} completely unchanged, while injecting negative reactive power $Q_{r,C} = -U^2 C\omega$ and zero active power ($P_C = 0$).

Execution:

(a) Capacitor powers $(P_C, Q_{r,C})$ and total active power P_{tot}: An ideal capacitor across RMS voltage U draws RMS current $I_C = U/|X_C| = C\omega U$ at phase angle $\varphi_C = -\pi/2$ ($\cos\varphi_C = 0$, $\sin\varphi_C = -1$):

$$P_C = U\,I_C\cos\left(-\frac{\pi}{2}\right) = 0\,\text{W},\qquad Q_{r,C} = U\,I_C\sin\left(-\frac{\pi}{2}\right) = -U\,(C\omega U) = -U^2 C\omega < 0.$$

By Boucherot's active power conservation, closing K **leaves the total active power drawn from the bus unchanged**:

$$P_{\text{tot}} = P + P_C = 864 + 0 = 864\,\text{W}.$$

(b) Capacitance C required to achieve unity power factor ($\cos\varphi' = 1$): Unity power factor ($\cos\varphi' = 1 \iff \varphi' = 0° \iff \tan\varphi' = 0$) requires zero total reactive power ($Q_{r,\text{tot}} = 0$):

$$Q_{r,\text{tot}} = Q_{r1} + Q_{r,C} = Q_{r1} - U^2 C\omega = 0 \implies C = \frac{Q_{r1}}{U^2\,\omega}.$$

Substituting $Q_{r1} = 1152\,\text{var}$, $U = 240\,\text{V}$ ($U^2 = 57\,600\,\text{V}^2$), and $\omega = 100\pi\,\text{rad}\,\text{s}^{-1}$:

$$C = \frac{1152}{240^2 \times 100\pi} = \frac{1152}{5\,760\,000\,\pi} = \frac{2\times10^{-4}}{\pi}\,\text{F} \approx 6.3662\times10^{-5}\,\text{F} \approx 63.66\,\mu\text{F}.$$

(General industrial compensation formula check: $C = \dfrac{P(\tan\varphi_1 - \tan\varphi')}{U^2\omega} = \dfrac{864(4/3 - 0)}{57\,600\times100\pi} = \dfrac{1152}{57\,600\times100\pi} \approx 63.66\,\mu\text{F}.$*)*

(c) New RMS line current I'_{line} and percentage reduction: From $P_{\text{tot}} = U\, I'_{\text{line}} \cos\varphi'$ with $P_{\text{tot}} = 864\,\text{W}$, $U = 240\,\text{V}$, and $\cos\varphi' = 1$:

$$I'_{\text{line}} = \frac{P_{\text{tot}}}{U\cos\varphi'} = \frac{864}{240\times 1} = 3.60\,\text{A}.$$

Relative reduction in RMS line current compared with $I_1 = 6.0\,\text{A}$:

$$\frac{I_1 - I'_{\text{line}}}{I_1} = \frac{6.0 - 3.60}{6.0} = \frac{2.40}{6.0} = 0.40 = 40\%.$$

$$P_C = 0\,\text{W},\quad Q_{r,C} = -U^2C\omega,\quad P_{\text{tot}} = 864\,\text{W},\quad C \approx 63.66\,\mu\text{F},\quad I'_{\text{line}} = 3.60\,\text{A}\ (-40\%)$$

✓ **Sanity Check:**
Notice the exact ratio $I'_{\text{line}}/I_1 = 3.60/6.0 = 0.60 = \cos\varphi_1$: raising the power factor from $\cos\varphi_1$ to 1 at constant active power P multiplies the supply line current by $\cos\varphi_1$ ($1 - \cos\varphi_1 = 40\%$ current drop) and cuts upstream feeder Joule losses $R_{\text{feeder}}I^2_{\text{line}}$ by $1 - 0.60^2 = 64\%$, while the coil itself still carries its full rated 6.0 A.

Exercise 10

Configuration Summary: *High-voltage AC power transmission ($P = 120\,\text{MW}$, $\cos\varphi = 1$, $20\,\text{kV} \rightarrow 400\,\text{kV}$), 4-wire low-voltage distribution ($U_{\text{LV}} = 380\,\text{V}$), and equal-loss conductor copper mass comparison between 2-wire single-phase and 3-wire balanced three-phase transmission.*

1) (a) Express the Joule power loss P_J per conductor in terms of R_{line} and I, and explain why stepping up U lowers line losses at fixed P. (b) By what factor do line Joule losses decrease when U is stepped up from 20 kV to EHV = 400 kV? ★☆☆☆☆

Method ***Inverse-Square Voltage Scaling of Transmission Line Joule Losses ($P_J \propto P^2/U^2$)***

Why we do this (Way of Thinking):
Students occasionally misapply $P_J = U^2/R_{\text{line}}$ using the *grid* voltage U, wrongly concluding that raising U increases line losses. In reality, U is the line-to-load transmission voltage whereas the small voltage drop across the conductor itself is $\Delta U_{\text{line}} = R_{\text{line}}I \ll U$, so one must evaluate $P_J = R_{\text{line}}I^2$ with I fixed by the transmitted power P.

Execution:

(a) Conductor Joule loss expression and physical scaling: Each conductor of resistance R_{line} carrying RMS current I dissipates Joule thermal power

$$P_J = R_{\text{line}}\, I^2.$$

At fixed transmitted active power P and unity power factor ($\cos\varphi = 1$), $I = \dfrac{P}{U}$ (single-phase) or $I = \dfrac{P}{\sqrt{3}\,U}$ (balanced three-phase); in both cases $I \propto \dfrac{P}{U}$, so

$$P_J \propto R_{\text{line}}\,\frac{P^2}{U^2}.$$

Stepping up the transmission RMS voltage U via a step-up transformer reduces the required line current I inversely with U and therefore lowers line Joule losses **inversely with U^2**.

(b) Loss reduction factor from $U_1 = 20\,\text{kV}$ to $U_2 = 400\,\text{kV}$:

$$\frac{P_J(20\,\text{kV})}{P_J(400\,\text{kV})} = \left(\frac{U_2}{U_1}\right)^2 = \left(\frac{400\,\text{kV}}{20\,\text{kV}}\right)^2 = 20^2 = 400.$$

$$P_J = R_{\text{line}}\, I^2 \propto \frac{R_{\text{line}}\, P^2}{U^2}, \qquad \frac{P_J(20\,\text{kV})}{P_J(400\,\text{kV})} = \left(\frac{400}{20}\right)^2 = 400$$

► **Common Pitfall:**
Never substitute the grid voltage $U = 400\,\text{kV}$ into U^2/R_{line}: 400 kV is the potential difference between phases (across the distant receiving substation), *not* the longitudinal ohmic drop $\Delta U_{\text{line}} = R_{\text{line}} I$ between the two ends of one wire.

2) Identify the four conductors of a 4-wire low-voltage distribution feeder ($U_{\text{LV}} = 380\,\text{V}$) and calculate the corresponding phase-to-neutral RMS voltage V_{LV} supplied to single-phase loads. ★☆☆☆☆

Method ***Balanced Star (Wye) Three-Phase Relation $U = \sqrt{3}\,V$***

Why we do this (Way of Thinking):
In a balanced three-phase four-wire low-voltage network, the three phase-to-neutral voltages $(\vec{V}_1, \vec{V}_2, \vec{V}_3)$ have equal RMS magnitude V_{LV} and are displaced by 120° ($2\pi/3$ rad). By the Fresnel law of cosines on $\vec{U}_{12} = \vec{V}_1 - \vec{V}_2$, the line-to-line voltage satisfies $U_{\text{LV}} = \sqrt{V^2 + V^2 - 2V^2\cos 120°} = \sqrt{3}\,V_{\text{LV}}$.

Execution: A 4-wire low-voltage distribution feeder consists of **3 active phase conductors** (L_1, L_2, L_3, displaced in phase by 120°) plus **1 neutral conductor** (N). Dividing the line-to-line RMS voltage $U_{\text{LV}} = 380\,\text{V}$ by $\sqrt{3}$ gives the phase-to-neutral RMS voltage:

$$V_{\text{LV}} = \frac{U_{\text{LV}}}{\sqrt{3}} = \frac{380\,\text{V}}{\sqrt{3}} \approx 219.39\,\text{V} \approx 220\,\text{V}.$$

4 conductors: 3 phase wires (L_1, L_2, L_3) + 1 neutral wire (N); $\qquad V_{\text{LV}} = \dfrac{380}{\sqrt{3}} \approx 220\,\text{V}$

✓ **Sanity Check:**
Under the harmonized IEC nominal standard $U_{\text{LV}} = 400\,\text{V}$, the exact same ratio yields $V_{\text{LV}} = 400/\sqrt{3} \approx 230.9\,\text{V} \approx 230\,\text{V}$, corresponding to the familiar 220/380 V and 230/400 V residential/industrial ratings.

3) Imposing identical total line Joule losses ($P_{J,1} = P_{J,3}$) between System 1 (2-wire single-phase, area S_1, $I_1 = P/U$) and System 3 (3-wire balanced three-phase, area S_3, $I_3 = P/(\sqrt{3}\,U)$) at identical (P, ℓ, U, ρ) and $\cos\varphi = 1$, determine S_1/S_3, deduce the total copper mass ratio M_1/M_3, and state the fractional extra copper mass needed by single-phase transmission. ★★★☆☆

Method ***Equal-Efficiency Conductor Cross-Section Sizing and Volumetric Copper Mass Ratio***

Why we do this (Way of Thinking):
Although a three-phase line uses 3 wires instead of 2, each wire carries only $I_3 = P/(\sqrt{3}\,U)$ instead of $I_1 = P/U$, cutting I^2 per wire by a factor of 3. Expressing the total Joule loss across *all* wires of each system ($2R_1 I_1^2$ vs. $3R_3 I_3^2$) with $R_k = \rho\ell/S_k$ fixes the wire cross-section ratio S_1/S_3, after which multiplying by the wire counts ($2\ell S_1$ vs. $3\ell S_3$) and copper mass density ρ_{Cu} yields the total copper mass ratio M_1/M_3.

Execution:

Step 1 — Total line Joule losses in System 1 and System 3: With individual conductor resistances $R_{\text{wire},1} =$

$\frac{\rho\ell}{S_1}$ and $R_{\text{wire},3} = \frac{\rho\ell}{S_3}$:

$$P_{J,1} = 2\,R_{\text{wire},1}\,I_1^2 = 2\left(\frac{\rho\ell}{S_1}\right)\left(\frac{P}{U}\right)^2 = \frac{2\,\rho\ell P^2}{S_1\,U^2},$$
$$P_{J,3} = 3\,R_{\text{wire},3}\,I_3^2 = 3\left(\frac{\rho\ell}{S_3}\right)\left(\frac{P}{\sqrt{3}\,U}\right)^2 = 3\left(\frac{\rho\ell}{S_3}\right)\frac{P^2}{3\,U^2} = \frac{\rho\ell P^2}{S_3\,U^2}.$$

Step 2 — Cross-section ratio S_1/S_3 under equal total Joule loss ($P_{J,1} = P_{J,3}$): Equating $P_{J,1} = P_{J,3}$ and canceling the common nonzero factor $\rho\ell P^2/U^2$:

$$\frac{2}{S_1} = \frac{1}{S_3} \implies \frac{S_1}{S_3} = 2 \quad (S_1 = 2\,S_3).$$

Step 3 — Total conductor copper mass ratio M_1/M_3 and fractional extra copper mass: For uniform copper mass density ρ_{Cu}, total conductor mass equals $\rho_{\text{Cu}} \times$ (number of wires) $\times\,\ell \times S$:

$$M_1 = \rho_{\text{Cu}}(2\,\ell\,S_1), \qquad M_3 = \rho_{\text{Cu}}(3\,\ell\,S_3).$$

Sub **Substitute $S_1 = 2\,S_3$ into M_1/M_3:**

$$\frac{M_1}{M_3} = \frac{\rho_{\text{Cu}}\,(2\,\ell\,S_1)}{\rho_{\text{Cu}}\,(3\,\ell\,S_3)} = \frac{2\,(2\,S_3)}{3\,S_3} = \frac{4}{3} \approx 1.333.$$

Writing $\frac{M_1}{M_3} = 1 + \frac{1}{3} \iff \frac{M_1 - M_3}{M_3} = +\frac{1}{3}$, single-phase 2-wire transmission requires $+\frac{1}{3}$ (+33.3%) **more copper mass** than balanced 3-wire three-phase transmission at identical power, voltage, distance, and line losses (equivalently, three-phase transmission saves $\frac{M_1-M_3}{M_1} = \frac{1}{4} = 25\%$ of the single-phase copper mass).

$$\frac{S_1}{S_3} = 2, \qquad \frac{M_1}{M_3} = \frac{4}{3} = 1 + \frac{1}{3} \quad \left(+\frac{1}{3} \approx +33.3\%\text{ extra copper mass in single-phase}\right)$$

✓ **Sanity Check:**
Notice why no neutral wire is needed in System 3: in a balanced three-phase load, the three line currents have equal RMS magnitude I_3 and 120° phase spacing, so their instantaneous sum vanishes identically ($i_a(t) + i_b(t) + i_c(t) = 0$), eliminating the return conductor that accounts for half the copper of a single-phase line.