Epsilite

Taylor expansion & limited development

Part I: Direct Computation of Limited Developments

These exercises test the ability to meticulously combine DLs through composition, integration, and division according to increasing powers while carefully managing truncation errors.

Exercise 1.1: Division According to Increasing Powers

Using the algorithm of division according to increasing powers, find the limited development at order 3 centered at 0 of the rational function:

$$f(x) = \frac{x^2 + 1}{x^3 - 1}$$

Hint: Eliminate the lowest-degree terms sequentially, rather than the highest, until reaching the desired order.

Exercise 1.2: Integration and Function Composition

Let $f(x) = \arctan(x)$ and $g(x) = \frac{\sin(x) - x}{\arctan(x) - x}$.

1. By expanding its derivative $f'(x) = \frac{1}{1+x^2}$, determine the DL of $f(x)$ at order 5 in the neighborhood of 0.
2. Deduce the DL of $g(x)$ at order 2 in the neighborhood of 0.

Exercise 1.3: Nested Composition and Algebraic Recentering

Calculate the limited development of $h(x) = (\cos x)^{\sin x}$ at order 5 in the neighborhood of 0. **Hint:** Rewrite the function using the exponential-logarithmic identity $h(x) = \exp(\sin x \ln(\cos x))$. Because $\cos(0) = 1$, you must recenter the inner function $u(x) = \cos x - 1$ to properly compose with $\ln(1 + u)$.

Part II: Resolution of Indeterminate Limits

These limits feature severe indeterminate forms where L'Hôpital's rule is inefficient. You must substitute functions with their polynomial equivalents and isolate the dominant term.

Exercise 2.1: Severe $0/0$ Form

Evaluate the following limit by identifying the first non-zero order of the expansions:

$$\lim_{x \to 0} \left(\frac{1}{\ln(1+x)} - \frac{1}{\tan x} \right)$$

Note: This difference of fractions requires a common denominator and a DL to order 2 for the numerator to properly evaluate the indeterminate form.

Exercise 2.2: Nested Transcendental $0/0$ Form

Determine the limit:

$$\lim_{x \to 0} \frac{e^{\sin x} - e^{\tan x}}{\sin x - \tan x}$$

Hint: Expand the denominator to order 3. For the numerator, factor out $e^{\sin x}$ or expand $e^{\sin x}$ and $e^{\tan x}$ to order 3 to resolve the cancellation.

Exercise 2.3: 1^∞ Indeterminate Form at Infinity

Calculate the limit:

$$\lim_{x \to +\infty} \left(\frac{2x-1}{2x+1} \right)^{2x}$$

Hint: Use the transformation $t = \frac{1}{2x}$ to map the limit from infinity to zero, then apply the exponential-logarithmic form.

Exercise 2.4: Sub-dominant Asymptotic Behavior

Evaluate the following limit:

$$\lim_{x \to +\infty} x \left(\left(1 + \frac{1}{x} \right)^x - e \right)$$

Hint: Set $X = 1/x$. You will need to expand $\ln(1+X)$ to order 2 to capture the sub-dominant term that remains after the e cancels.

Part III: Geometric Applications

These problems apply local polynomial approximations to deduce global or geometric properties of curves, such as tangents, asymptotes, and inflection points.

Exercise 3.1: Complete Local and Asymptotic Study

Let f be the function defined on $]-1,1[\cup]1,+\infty[$ by:

$$f(x) = (x^2 - 1)\ln\left|\frac{1-x}{1+x}\right|$$

1. **Local Behavior:** Find the DL of $f(x)$ at order 3 in a neighborhood of 0. Deduce the Cartesian equation of the tangent line (T) at $x = 0$ and determine the position of the curve relative to (T).
2. **Asymptotic Behavior:** Using a change of variable $t = 1/x$, find the asymptotic expansion of f as $x \to +\infty$. Prove that the graph of f admits an oblique asymptote (A), give its Cartesian equation, and determine the position of the curve relative to (A) based on the sign of the remaining dominant term.

Exercise 3.2: Parameter Optimization

Determine the real numbers a and b such that the principal part of the limited development at 0 of the function:

$$g(x) = \frac{\cos x - 1 + ax^2}{1 + bx^2}$$

is of the highest possible valuation (order).
Hint: Perform an abstract expansion of the quotient to order 6 and set the coefficients of x^2 and x^4 to zero.

Part IV: Combinatoric Symmetry (Bonus)

This acts as an algebraic stress test. You must use the symmetry of the functions to drastically reduce the required calculations.

Exercise 4.1: The Odd Parity Shortcut

Calculate the limited development at order 7 in the neighborhood of 0 for the highly nested function:

$$k(x) = \arcsin(\arctan x) - \arctan(\arcsin x)$$

Hint: Both $\arcsin(x)$ and $\arctan(x)$ are odd functions. Because the composition of odd functions is strictly odd, their Taylor polynomials will contain no even-powered coefficients ($a_2 = a_4 = a_6 = 0$). Furthermore, the first-order terms (x) and third-order terms will cancel out upon subtraction. Leverage this odd parity to minimize the massive polynomial expansion required, focusing only on the calculation of the coefficients for x^5 and x^7.