LD Applications Set

Functions study, inequalities, and Integration

Part I: Function study

Exercise 1.1:

Let $f(x) = (1+\tanh(x))^{\frac{1}{\cos(x)}}$.

1. Calculate the limited development at order 3 of f in the neighborhood of 0.
2. Find the exact values of $f'(0)$, $f''(0)$, and $f'''(0)$.
3. Deduce the Cartesian equation of the tangent line (T) and its position relative to the curve (C_f).
4. What can you say about the point $(0, f(0))$?
5. Deduce the limit:
$$\lim_{x\to 0} \frac{f(x) - e^x + \frac{x^2}{2}}{x - \sin(x)}$$

Exercise 1.2:

Let $g(x) = \arctan\left(\sqrt{\frac{\cos(2x)}{\cosh(4x)}}\right)$.

1. Calculate the limited development of g at order 4 in the neighborhood of 0.
2. Find the values of $g'(0)$, $g''(0)$, $g'''(0)$, and $g^{(4)}(0)$.
3. What can you say about the point $(0, g(0))$?
4. Evaluate the limit:
$$\lim_{x\to 0} \frac{g(x) - \frac{\pi}{4}}{1 - \cosh(3x)}$$

Exercise 1.3:

Let the function h be defined for $x > 0$ by:

$$h(x) = \frac{x^2+1}{x} e^{\frac{1}{x}}$$

1. By using the change of variable $t = \frac{1}{x}$, calculate the limited development of $h(x)$ as $x \to +\infty$ up to the term in $\frac{1}{x^2}$.
2. Deduce the Cartesian equation of the asymptote (D) to the curve (C_h) as $x \to +\infty$.
3. Determine the relative position of (C_h) with respect to (D) in the neighborhood of $+\infty$.

Part II: Analytical Inequalities

Use the Taylor-Lagrange formula or local developments to prove the following bounds.

Exercise 2.1:

Prove that for all $x \in \mathbb{R}$:

$$\cos(x) \geq 1 - \frac{x^2}{2}$$

Exercise 2.2:

Using the Taylor-Lagrange inequality, prove that for all $x \geq 0$:

$$x + \frac{x^3}{6} \leq \sinh(x) \leq x + \frac{x^3}{6}\cosh(x)$$

Exercise 2.3:

Determine the neighborhood $]0, \alpha[$ for which the following inequality holds strictly true:

$$\cos(x) < \left(\frac{\sin x}{x}\right)^3$$

Part III: Generalized Expansions & Integration

Exercise 3.1:

Calculate the generalized limited development at order 3 (in fractional powers $x^{k/2}$) in the neighborhood of 0^+ for:

$$m(x) = \frac{\cos(\sqrt{x}) - e^{-x/2}}{x^{3/2}}$$

Exercise 3.2:

Let $f(x) = \ln(x + \sqrt{1 + x^2})$.

1. Calculate the exact derivative $f'(x)$ and simplify it.
2. Calculate the limited development of $f'(x)$ at order 4 in the neighborhood of 0.
3. Deduce the limited development of the original function $f(x)$ at order 5 in the neighborhood of 0.

Part IV: Applied Embedded Systems (bonus)

Exercise 4.1: Kinematic Control Law Optimization

To minimize calculation time on an embedded microcontroller, a non-linear steering correction algorithm $\omega(\theta)$ for a differential drive robot must be approximated. The exact law is:

$$\omega(\theta) = \frac{\tan\theta}{\sqrt{1 - \frac{1}{2}\sin^2\theta}}$$

Calculate the limited development of $\omega(\theta)$ at order 5 in the neighborhood of $\theta = 0$ to provide a highly accurate polynomial replacement for the control loop.