Integrals

The Second Part of the Integrals set That covers IBP and Changing Variables Integrals Tricks

Category 1:

1. $I_1 = \int x \tan^2(x)\, dx$
2. $I_2 = \int \cos(\ln x)\, dx$
3. $I_3 = \int_0^{\pi} e^{-x} \sin(x)\, dx$
4. $I_4 = \int \frac{x \ln(x + \sqrt{x^2+1})}{\sqrt{x^2+1}}\, dx$
5. $I_5 = \int \frac{x^2 e^x}{(x+2)^2}\, dx$
6. $I_6 = \int \frac{x^2}{(x \sin x + \cos x)^2}\, dx$

Category 2:

7. $I_7 = \int \frac{1}{\sqrt{x} + \sqrt[3]{x}}\, dx$
8. $I_8 = \int \frac{1}{x^2\sqrt{x^2+4}}\, dx$
9. $I_9 = \int \frac{\sqrt{x}}{1+x^3}\, dx$
10. $I_{10} = \int \frac{\cos 2x}{\sin^4 x + \cos^4 x}\, dx$
11. $I_{11} = \int \frac{x^2}{(x+1)\sqrt{x-1}}\, dx$

Category 3:

12. $I_{12} = \int_0^1 \frac{x}{x^3+x^2+x+1}\, dx$
13. $I_{13} = \int \frac{x^2+1}{x^4+1}\, dx$
14. $I_{14} = \int \frac{1}{x^3+1}\, dx$
15. $I_{15} = \int \frac{3x+1}{(x-1)^2(x^2+1)}\, dx$

Category 4:

16. $I_{16} = \int \frac{\sin^3 x}{2 + \cos x} \, dx$
17. $I_{17} = \int \frac{\cos x}{\sin^2 x - 5 \sin x + 6} \, dx$
18. $I_{18} = \int \frac{1}{1 + \sin^2 x} \, dx$
19. $I_{19} = \int \frac{\sin x + \cos x}{\sqrt{\sin 2x}} \, dx$

Category 5:

20. $I_{20} = \int \frac{x^3}{\sqrt{x^2 + 1}} \, dx$
21. $I_{21} = \int \frac{dx}{x\sqrt{6 - x - x^2}}$
22. $I_{22} = \int \frac{\sqrt{x^2 - 1}}{x} \, dx$

Category 6:

23. $I_{23} = \int \frac{1}{2 + \cos x} \, dx$
24. $I_{24} = \int \frac{1}{\sin x + \cos x + 1} \, dx$

Category 7:

25. $I_{25} = \int (\sin x \cos x) e^{\sin x} \, dx$
26. $I_{26} = \int_0^{\pi/4} \ln(1 + \tan x) \, dx$
27. $I_{27} = \int_0^1 \frac{\ln(x + 1)}{x^2 + 1} \, dx$
28. $I_{28} = \int_0^{\pi/2} \ln(2 \cos x) \, dx$
29. $I_{29} = \int \frac{3}{x^4 + x} \, dx$
30. $I_{30} = \int \frac{1}{(1 + x)\sqrt{1 + x + x^2}} \, dx$
31. $I_{31} = \int_0^{\pi/2} \frac{\sin^3 x}{\sin^3 x + \cos^3 x} \, dx$
32. $I_{32} = \int \frac{1}{x}\sqrt{\frac{1 - x}{1 + x}} \, dx$

Category 8: theoretical problems

33. **Problem I :**
Let f be a continuous function on $[a, b]$.
   (a) Prove that $\displaystyle\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$.
   (b) Let g be a continuous, strictly positive function on $[0, \pi/2]$, prove that:
$$\int_0^{\pi/2} \frac{g(\sin x)}{g(\sin x) + g(\cos x)}\,dx = \frac{\pi}{4}$$
   (c) Deduce the exact value of $\int_0^1 \frac{1}{\sqrt{1-x^2}\left(1+e^{1-2x^2}\right)}\,dx$.

34. **Problem II :**
Consider the integral sequence defined by $W_n = \displaystyle\int_1^e (\ln x)^n\,dx$ for $n \in \mathbb{N}$.
   (a) Establish the reduction formula: $W_n = e - nW_{n-1}$.
   (b) Prove that for all $n \in \mathbb{N}$, $W_n \geq 0$, then deduce the monotony of the sequence (W_n).
   (c) Using the monotonicity and the reduction formula, prove that $\displaystyle\lim_{n\to+\infty} W_n = 0$.

35. **Problem III :**
To calculate integrals like $e^{-x}\sin x$ or $\cos(\ln x)$, we often integrate by parts twice. We can instead solve them as a system. Let $\alpha, \beta \in \mathbb{R}^*$. Define:
$$J = \int e^{\alpha x}\cos(\beta x)\,dx \quad \text{and} \quad K = \int e^{\alpha x}\sin(\beta x)\,dx$$
   (a) Using Integration by Parts just **once** on each, construct a linear system of two equations relating J and K.
   (b) Solve the system to find the explicit formulas for J and K.
   (c) Apply your generalized formula to directly write the primitive of $I_2 = \int \cos(\ln x)\,dx$.

36. **Problem IV :**
Consider the generalized family of integrals $A_n = \displaystyle\int \frac{1}{x\sqrt{1+x^n}}\,dx$ where $n \in \mathbb{N}^*$.
   (a) Prove that :
$$A_n = \frac{2}{n}\int \frac{1}{t^2-1}\,dt$$
   (b) Deduce the final primitive $A_n(x)$ in terms of x.
   (c) Prove that $\displaystyle\lim_{n\to+\infty}\left(\int_1^2 \frac{1}{x\sqrt{1+x^n}}\,dx\right) = 0$.

37. **Problem V :**
Consider the sequence $J_n = \int_0^{\pi/4} \tan^n(x)\,dx$ for $n \in \mathbb{N}$.

   (a) Show that for all $n \in \mathbb{N}$, $J_n + J_{n+2} = \dfrac{1}{n+1}$.
   (b) Study the sign and the monotony of (J_n) then deduce its nature.
   (c) By using the relation from part (a), show that $\lim_{n\to+\infty} J_n = 0$.
   (d) Using the monotonicity of (J_n) and the formula from question (a), establish the double inequality:
   $$\frac{1}{2(n+1)} \leq J_n \leq \frac{1}{2(n-1)} \quad \text{for } n \geq 2$$
   (e) Conclude by giving a simple equivalent for J_n as $n \to +\infty$.

38. **Problem VI :**
Consider the sequence $I_n = \int_0^{\pi/2} \sin^n(x)\,dx$ for $n \in \mathbb{N}$.

   (a) Establish the recurrence relation: $I_n = \dfrac{n-1}{n} I_{n-2}$ for $n \geq 2$.
   (b) Study the sign and the monotony of (I_n).
   (c) By evaluating nI_nI_{n-1},calculate the product nI_nI_{n-1}.
   (d) Prove that $1 \leq \dfrac{I_{n-2}}{I_n} \leq \dfrac{n}{n-1}$ then deduce $\lim_{n\to+\infty} \dfrac{I_{n-1}}{I_n}$, and find an equivalent for I_n as $n \to +\infty$.

39. **Problem VII :**

   (a) Let $f : [a,b] \to \mathbb{R}$ be a continuous function, differentiable on $]a,b[$, such that:
   $$f(a) = f(b) = 0.$$
   Prove that for any real number $\lambda \in \mathbb{R}$, there exists $c \in]a,b[$ such that $f'(c)+\lambda f(c) = 0$.

   (b) Let $g : [0,1] \to \mathbb{R}$ be a continuous function such that $\int_0^1 g(x)\,dx = \dfrac{1}{2}$. By defining the integral function $G(x) = \int_0^x (g(t) - t)\,dt$, prove that there exists $\alpha \in]0,1[$ such that $g(\alpha) = \alpha$.

   (c) Let $u, v : [a,b] \to \mathbb{R}$ be two continuous functions. Assume u is continuously differentiable (C^1), strictly positive, and strictly decreasing.
   We define the area function $V(x) = \int_a^x v(t)\,dt$.
   Let m and M be the minimum and maximum of V on $[a,b]$.
      i. Prove that:
      $$\int_a^b u(x)v(x)\,dx = u(b)V(b) - \int_a^b u'(x)V(x)\,dx$$
      ii. Bound the integral term $-\int_a^b u'(x)V(x)\,dx$ using m and M.
      iii. Deduce the rigorous enclosure:
      $$m \cdot u(a) \leq \int_a^b u(x)v(x)\,dx \leq M \cdot u(a)$$

iv. Prove the existence of $c \in [a, b]$ such that:

$$\int_a^b u(x)v(x)\,dx = u(a)\int_a^c v(x)\,dx$$

40. **Problem VIII :**

Consider the definite integral $\Omega = \displaystyle\int_0^{\pi/2} \frac{1}{(2\sin x + \cos x + 3)^2}\,dx.$

(a) Demonstrate algebraically that Ω transforms into:

$$\Omega = \frac{1}{2}\int_0^1 \frac{t^2+1}{(t^2+2t+2)^2}\,dt$$

(b) Write the quadratic polynomial $t^2 + 2t + 2$ in its canonical form.

(c) Show that the integral transforms into:

$$\Omega = \frac{1}{2}\int_1^2 \frac{u^2-2u+2}{(u^2+1)^2}\,du$$

(d) compute the exact, simplified final numerical value of Ω.

Bonus part:

this part has some beautiful integrals out of our program, if you have time and you want to have some fun, "try" to solve it :)

1- $I_{34} = \displaystyle\int_{-\infty}^{+\infty} \frac{\sin x}{x}\,dx$

2- $I_{35} = \displaystyle\int_0^1 x\left\lfloor \frac{1}{x} \right\rfloor dx$

3- $I_{36} = \displaystyle\int_0^1 x^x\,dx$