Ordinary Differential Equations

Collection of analytical problems covering ODEs with resonance, linearization, and order reduction.

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Category 1: Separable & Homogeneous Equations

1. Solve the non-linear equation $\sin(y)y' = \frac{x\cos(y)}{1-x^2}$ on $x \in]-1,1[$.

2. Solve the Cauchy Problem: find the general solution to $y' = \frac{x^2+3y^2}{2xy}$ with $y(1) = 2$.

Category 2: First-Order Linear Equations

3. Solve the equation $(x^2-1)y' + 2xy = \frac{1}{(x^2-1)^2}$ for $x > 1$.

4. Solve the Cauchy problem: $\cos(x)y' + \sin(x)y = e^{-x}\cos^2(x)$ with $y(0) = 1$ on the interval $]-\frac{\pi}{2}, \frac{\pi}{2}[$.

Category 3: Bernoulli Linearizations & Riccati Equations

5. Solve the Bernoulli equation: $y' + \frac{1}{x\ln(x)}y = x^2\ln^2(x)y^3$ for $x > 1$.

6. Consider the Riccati equation $(R_1) : y' = \frac{1}{2}y^2 + \frac{1}{2x^2}$.

(a) Find a particular solution of the form $y_p(x) = \frac{c}{x}$ by identifying the constant c.

(b) By making the substitution $y(x) = y_p(x) + \frac{1}{z(x)}$, find the general solution of (R_1).

7. Consider the Riccati equation $(R_2) : y' = y^2 - \left(2 + \frac{2}{x}\right)y + \left(1 + \frac{2}{x}\right)$. Find the constant particular solution $y_p \in \mathbb{R}$, and fully solve the equation.

Category 4: Constant Coefficients & Resonance

8. Find the general solution to $y'' - 3y' + == e^{2x}(1+x) + \sin(x)$.

9. Find the general solution to $y'' + 2y' + 5y = e^{-x}\cos(2x)$.

Category 5: Variation of Parameters (2nd Order)

10. Find the general solution to $y'' + y = \frac{1}{\cos^3(x)}$ on $]-\frac{\pi}{2}, \frac{\pi}{2}[$.

11. Consider the equation $y'' - 4y' + 4y = xe^{2x}\ln(x)$ for $x > 0$.

(a) Justify briefly why the method of undetermined coefficients cannot be applied to this equation.

(b) Solve the equation using the method of Variation of Parameters.

Category 6: Variable Coefficients & Order Reduction

12. Solve $xy'' + y' = (y')^2$ using the substitution $z = y'$.

13. Find the general solution to $xy'' - (x+1)y' + y = 0$ given that $y_1(x) = e^x$ is a particular solution.

14. Solve the Euler equation $x^2y'' - xy' + y = x\ln(x)$ using the substitution $x = e^t$.

Category 7: Theoretical Synthesis **(not recommended)**

15. Let $(E) : y'' + a(x)y' + b(x)y = 0$ be a homogeneous linear differential equation. Suppose $y_1(x)$ and $y_2(x)$ are two solutions of (E). Define the Wronskian:

$$W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$$

(a) By differentiating $W(x)$ with respect to x and substituting y_1'' and y_2'' from (E), prove that $W(x)$ satisfies the first-order separable differential equation: $W' + a(x)W = 0$.

(b) Deduce that if $W(x_0) \neq 0$ at some point x_0, then $W(x)$ is never zero on the interval of definition.

Bonus:

16. Consider the second-order differential equation: *Optional*

$$yy'' + (y')^2 = 1$$

(a) Let $u(x) = (y(x))^2$. Compute $u'(x)$ and $u''(x)$ in terms of y, y' and y''.

(b) Substitute into the original equation and show that it reduces to a second-order linear equation with constant coefficients in $u(x)$.

(c) Solve the resulting linear equation for $u(x)$.

(d) Deduce the general implicit solution for $y(x)$, and identify the geometric nature of the solution curves in the xy-plane.