Taylor Expansion & Limited Development

Correction

Part I: Direct Computation of Limited Developments

Exercise 1.1: Division According to Increasing Powers

Answer:

$$f(x) = -1 - x^2 - x^3 + o(x^3)$$

Steps:

$$\begin{aligned} 1 + x^2 &= (-1)(-1 + x^3) + (x^2 + x^3) \\ x^2 + x^3 &= (-x^2)(-1 + x^3) + x^3 + x^5 \\ x^3 &= (-x^3)(-1 + x^3) + x^6 \\ \frac{1 + x^2}{-1 + x^3} &= -1 - x^2 - x^3 + o(x^3) \end{aligned}$$

Exercise 1.2: Integration and Function Composition

Answer 1:

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} + o(x^5)$$

Answer 2:

$$g(x) = \frac{1}{2} + \frac{11x^2}{40} + o(x^2)$$

Steps:

$$f'(x) = \frac{1}{1 + x^2} = 1 - x^2 + x^4 + o(x^4)$$

$$f(x) = \int_0^x (1 - t^2 + t^4)dt = x - \frac{x^3}{3} + \frac{x^5}{5} + o(x^5)$$

$$g(x) = \frac{\sin(x) - x}{\arctan(x) - x}$$

$$\sin(x) - x = -\frac{x^3}{6} + \frac{x^5}{120} + o(x^5)$$

$$\arctan(x) - x = -\frac{x^3}{3} + \frac{x^5}{5} + o(x^5)$$

$$g(x) = \frac{-\frac{x^3}{6} + \frac{x^5}{120}}{-\frac{x^3}{3} + \frac{x^5}{5}} = \frac{1}{2}\frac{1 - \frac{x^2}{20}}{1 - \frac{3x^2}{5}}$$

$$\frac{1}{1 - \frac{3x^2}{5}} = 1 + \frac{3x^2}{5} + o(x^2)$$

$$g(x) = \frac{1}{2}\left(1 - \frac{x^2}{20}\right)\left(1 + \frac{3x^2}{5}\right) = \frac{1}{2}\left(1 + \left(\frac{3}{5} - \frac{1}{20}\right)x^2\right) = \frac{1}{2} + \frac{11x^2}{40} + o(x^2)$$

Exercise 1.3: Nested Composition and Algebraic Recentering

Answer:

$$h(x) = 1 - \frac{x^3}{2} + o(x^5)$$

Steps:

$$h(x) = \exp(\sin x \ln(\cos x))$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5)$$

$$u(x) = -\frac{x^2}{2} + \frac{x^4}{24}$$

$$\ln(1+u) = u - \frac{u^2}{2} = \left(-\frac{x^2}{2} + \frac{x^4}{24}\right) - \frac{1}{2}\left(-\frac{x^2}{2}\right)^2 + o(x^5) = -\frac{x^2}{2} - \frac{x^4}{12} + o(x^5)$$

$$\sin x = x - \frac{x^3}{6} + o(x^4)$$

$$\sin x \ln(\cos x) = \left(x - \frac{x^3}{6}\right)\left(-\frac{x^2}{2} - \frac{x^4}{12}\right) = -\frac{x^3}{2} - \frac{x^5}{12} + \frac{x^5}{12} + o(x^5) = -\frac{x^3}{2} + o(x^5)$$

$$h(x) = \exp\left(-\frac{x^3}{2} + o(x^5)\right) = 1 - \frac{x^3}{2} + o(x^5)$$

Part II: Resolution of Indeterminate Limits

Exercise 2.1: Severe $0/0$ Form

Answer:

$$\frac{1}{2}$$

Steps:

$$\lim_{x\to 0} \frac{\tan x - \ln(1+x)}{\tan x \ln(1+x)}$$

$$\text{Denominator: } (x + o(x))(x + o(x)) = x^2 + o(x^2)$$

$$\text{Numerator: } \left(x + \frac{x^3}{3} + o(x^2)\right) - \left(x - \frac{x^2}{2} + o(x^2)\right) = \frac{x^2}{2} + o(x^2)$$

$$\lim_{x\to 0} \frac{\frac{x^2}{2} + o(x^2)}{x^2 + o(x^2)} = \frac{1}{2}$$

Exercise 2.2: Nested Transcendental $0/0$ Form

Answer:

$$1$$

Steps:

$$\lim_{x\to 0} \frac{e^{\sin x}(1 - e^{\tan x - \sin x})}{\sin x - \tan x}$$

$$\tan x - \sin x \to 0 \text{ as } x \to 0$$

$$1 - e^{\tan x - \sin x} = -(\tan x - \sin x) + o(\tan x - \sin x)$$

$$\lim_{x\to 0} e^{\sin x} \frac{-(\tan x - \sin x)}{\sin x - \tan x} = \lim_{x\to 0} e^{\sin x}(1) = e^0 = 1$$

Exercise 2.3: 1^∞ Indeterminate Form at Infinity

Answer:

$$e^{-2}$$

Steps:

$$\lim_{x\to+\infty} \exp\left(2x \ln\left(\frac{2x-1}{2x+1}\right)\right)$$

$$t = \frac{1}{2x} \implies \lim_{t\to0^+} \exp\left(\frac{1}{t} \ln\left(\frac{1-t}{1+t}\right)\right)$$

$$\ln(1-t) - \ln(1+t) = \left(-t - \frac{t^2}{2} + o(t^2)\right) - \left(t - \frac{t^2}{2} + o(t^2)\right) = -2t + o(t^2)$$

$$\lim_{t\to0^+} \exp\left(\frac{-2t + o(t^2)}{t}\right) = \exp(-2) = e^{-2}$$

Exercise 2.4: Sub-dominant Asymptotic Behavior

Answer:

$$-\frac{e}{2}$$

Steps:

$$X = \frac{1}{x} \implies \lim_{X\to0^+} \frac{1}{X}\left(\exp\left(\frac{1}{X}\ln(1+X)\right) - e\right)$$

$$\ln(1+X) = X - \frac{X^2}{2} + \frac{X^3}{3} + o(X^3)$$

$$\frac{1}{X}\ln(1+X) = 1 - \frac{X}{2} + \frac{X^2}{3} + o(X^2)$$

$$\exp\left(1 - \frac{X}{2} + \frac{X^2}{3} + o(X^2)\right) = e \cdot \exp\left(-\frac{X}{2} + \frac{X^2}{3}\right)$$

$$e\left(1 + \left(-\frac{X}{2} + \frac{X^2}{3}\right) + \frac{1}{2}\left(-\frac{X}{2}\right)^2 + o(X^2)\right) = e\left(1 - \frac{X}{2} + \frac{11X^2}{24} + o(X^2)\right)$$

$$\lim_{X\to0^+} \frac{1}{X}\left(e - e\frac{X}{2} + o(X) - e\right) = \lim_{X\to0^+}\left(-\frac{e}{2} + o(1)\right) = -\frac{e}{2}$$

Part III: Geometric Applications

Exercise 3.1: Complete Local and Asymptotic Study

Answers:

1. **DL at 0:** $f(x) = 2x - \frac{4}{3}x^3 + o(x^3)$. Tangent $(T) : y = 2x$. Curve is above (T) for $x < 0$, below for $x > 0$.
2. **Asymptote:** $(A) : y = -2x$. Curve is above (A) for $x \to +\infty$.

Steps:

1. Local Behavior:

$$\ln\left(\frac{1-x}{1+x}\right) = -2\left(x + \frac{x^3}{3} + o(x^3)\right)$$

$$f(x) = (x^2-1)\left(-2x - \frac{2x^3}{3} + o(x^3)\right) = 2x - 2x^3 + \frac{2x^3}{3} + o(x^3) = 2x - \frac{4}{3}x^3 + o(x^3)$$

$$f(x) - y_{(T)} = -\frac{4}{3}x^3 \implies \operatorname{sgn}\left(-\frac{4}{3}x^3\right)$$

2. Asymptotic Behavior:

$$X = \frac{1}{x} \implies f(x) = \left(\frac{1}{X^2} - 1\right)\ln\left(\frac{X-1}{X+1}\right)$$

$$\ln\left(\frac{1-X}{1+X}\right) = -2\left(X + \frac{X^3}{3} + \frac{X^5}{5} + o(X^5)\right)$$

$$f(x) = \left(\frac{1-X^2}{X^2}\right)\left(-2X - \frac{2X^3}{3} + o(X^3)\right) = \left(\frac{1}{X^2} - 1\right)\left(-2X - \frac{2X^3}{3} + o(X^3)\right)$$

$$= -\frac{2}{X} - \frac{2X}{3} + 2X + o(X) = -2x + \frac{4}{3x} + o\left(\frac{1}{x}\right)$$

$$f(x) - y_{(A)} = \frac{4}{3x} > 0 \text{ as } x \to +\infty$$

Exercise 3.2: Parameter Optimization

Answer: No exact solution exists for the algebraic form exactly as written. (If derived as a standard Padé approximation $g(x) = \cos x - \frac{1+ax^2}{1+bx^2}$, $a = -\frac{5}{12}, b = \frac{1}{12}$).

Steps for given explicit function:

$$g(x) = \left((a - \frac{1}{2})x^2 + \frac{x^4}{24} + o(x^4)\right)(1 - bx^2 + o(x^2))$$

$$g(x) = (a - \frac{1}{2})x^2 + \left(\frac{1}{24} - b(a - \frac{1}{2})\right)x^4 + o(x^4)$$

$$a - \frac{1}{2} = 0 \implies a = \frac{1}{2}$$

$$\frac{1}{24} - b(0) = \frac{1}{24} \neq 0$$

Part IV: Combinatoric Symmetry (Bonus)

Exercise 4.1: The Odd Parity Shortcut

Answer:

$$k(x) = -\frac{x^7}{30} + o(x^7)$$

Steps:

$$A(x) = \arcsin\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}\right)$$

$$A(x) = \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}\right) + \frac{1}{6}\left(x - \frac{x^3}{3} + \frac{x^5}{5}\right)^3 + \frac{3}{40}\left(x - \frac{x^3}{3}\right)^5 + \frac{5}{112}x^7$$

$$A(x)_{x^5} = \frac{1}{5} + \frac{1}{6}(-1) + \frac{3}{40}(1) = \frac{13}{120}$$

$$A(x)_{x^7} = -\frac{1}{7} + \frac{1}{6}\left(3\left(\frac{1}{5}\right) + 3\left(-\frac{1}{3}\right)^2\right) + \frac{3}{40}\left(-5\left(\frac{1}{3}\right)\right) + \frac{5}{112} = -\frac{341}{5040}$$

$$B(x) = \arctan\left(x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112}\right)$$

$$B(x) = \left(x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112}\right) - \frac{1}{3}\left(x + \frac{x^3}{6} + \frac{3x^5}{40}\right)^3 + \frac{1}{5}\left(x + \frac{x^3}{6}\right)^5 - \frac{1}{7}x^7$$

$$B(x)_{x^5} = \frac{3}{40} - \frac{1}{3}\left(\frac{3}{6}\right) + \frac{1}{5}(1) = \frac{13}{120}$$

$$B(x)_{x^7} = \frac{5}{112} - \frac{1}{3}\left(3\left(\frac{3}{40}\right) + 3\left(\frac{1}{6}\right)^2\right) + \frac{1}{5}\left(5\left(\frac{1}{6}\right)\right) - \frac{1}{7} = -\frac{173}{5040}$$

$$k(x) = A(x)_{x^7} - B(x)_{x^7} = \left(-\frac{341}{5040} - \left(-\frac{173}{5040}\right)\right)x^7 = -\frac{168}{5040}x^7 = -\frac{x^7}{30}$$