



LD applications set Correction

Detailed Solutions and Derivations

Part I:

Exercise 1.1:

Answers:

1. $f(x) = 1 + x + \frac{x^3}{6} + o(x^3)$
2. $f'(0) = 1, f''(0) = 0, f'''(0) = 1$
3. $(T) : y = 1 + x$. Curve is above (T) for $x > 0$, below for $x < 0$.
4. The point $(0, 1)$ is an inflection point.
5. Limit is 0.

Rigorous Step-by-Step Derivations:

$$f(x) = \exp\left(\frac{1}{\cos x} \ln(1 + \tanh x)\right)$$

Step 1: Expansion of $\ln(1 + \tanh x)$

$$\tanh x = x - \frac{x^3}{3} + o(x^3)$$

Let $u = x - \frac{x^3}{3}$. We use the expansion $\ln(1 + u) = u - \frac{u^2}{2} + \frac{u^3}{3} + o(u^3)$. We must calculate the powers of u up to order 3:

$$\begin{aligned}u^2 &= \left(x - \frac{x^3}{3}\right)^2 = x^2 - 2(x)\left(\frac{x^3}{3}\right) + o(x^3) = x^2 + o(x^3) \\u^3 &= \left(x - \frac{x^3}{3}\right)^3 = x^3 + o(x^3)\end{aligned}$$

Substitute back into the logarithm expansion:

$$\begin{aligned}\ln(1 + \tanh x) &= \left(x - \frac{x^3}{3}\right) - \frac{1}{2}(x^2) + \frac{1}{3}(x^3) + o(x^3) \\&= x - \frac{x^2}{2} + \left(-\frac{1}{3} + \frac{1}{3}\right)x^3 + o(x^3) \\&= x - \frac{x^2}{2} + o(x^3)\end{aligned}$$

Step 2: Expansion of $\frac{1}{\cos x}$

$$\cos x = 1 - \frac{x^2}{2} + o(x^3)$$

Let $v = -\frac{x^2}{2}$. We use the expansion $\frac{1}{1+v} = 1 - v + v^2 + o(v^2)$.

$$\begin{aligned}\frac{1}{\cos x} &= 1 - \left(-\frac{x^2}{2}\right) + o(x^3) \\ &= 1 + \frac{x^2}{2} + o(x^3)\end{aligned}$$

Step 3: Product of the two expansions We multiply the results from Step 1 and Step 2:

$$\begin{aligned}\frac{1}{\cos x} \ln(1 + \tanh x) &= \left(1 + \frac{x^2}{2} + o(x^3)\right) \left(x - \frac{x^2}{2} + o(x^3)\right) \\ &= 1 \cdot \left(x - \frac{x^2}{2}\right) + \frac{x^2}{2} \cdot (x) + o(x^3) \\ &= x - \frac{x^2}{2} + \frac{x^3}{2} + o(x^3)\end{aligned}$$

Step 4: Composition with the Exponential Let $w = x - \frac{x^2}{2} + \frac{x^3}{2}$. We use $e^w = 1 + w + \frac{w^2}{2} + \frac{w^3}{6} + o(w^3)$. Calculate powers of w :

$$\begin{aligned}w^2 &= \left(x - \frac{x^2}{2} + \frac{x^3}{2}\right)^2 = x^2 + 2(x) \left(-\frac{x^2}{2}\right) + o(x^3) = x^2 - x^3 + o(x^3) \\ w^3 &= \left(x - \frac{x^2}{2} + \frac{x^3}{2}\right)^3 = x^3 + o(x^3)\end{aligned}$$

Substitute into the exponential expansion:

$$\begin{aligned}f(x) &= 1 + \left(x - \frac{x^2}{2} + \frac{x^3}{2}\right) + \frac{1}{2}(x^2 - x^3) + \frac{1}{6}(x^3) + o(x^3) \\ &= 1 + x + x^2 \left(-\frac{1}{2} + \frac{1}{2}\right) + x^3 \left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right) + o(x^3) \\ &= 1 + x + \frac{x^3}{6} + o(x^3)\end{aligned}$$

Step 5: Derivatives, Tangent, and Limit Using Maclaurin's identification: $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + o(x^3)$. Equating coefficients:

$$f(0) = 1, \quad f'(0) = 1, \quad \frac{f''(0)}{2} = 0 \implies f''(0) = 0, \quad \frac{f'''(0)}{6} = \frac{1}{6} \implies f'''(0) = 1$$

The tangent equation is $y = f(0) + f'(0)x = 1 + x$. Position is given by the sign of $f(x) - y_{(T)} = \frac{x^3}{6}$. It is positive for $x > 0$ (curve above tangent) and negative for $x < 0$ (curve below tangent). The change in concavity makes $(0, 1)$ an inflection point. For the limit:

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{f(x) - e^x + \frac{x^2}{2}}{x - \sin(x)} &= \lim_{x \rightarrow 0} \frac{\left(1 + x + \frac{x^3}{6}\right) - \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right) + \frac{x^2}{2} + o(x^3)}{x - \left(x - \frac{x^3}{6}\right) + o(x^3)} \\ &= \lim_{x \rightarrow 0} \frac{0 + o(x^3)}{\frac{x^3}{6} + o(x^3)} = 0\end{aligned}$$

Exercise 1.2:

Answers:

- $g(x) = \frac{\pi}{4} - \frac{5}{2}x^2 + 5x^4 + o(x^4)$
- $g'(0) = 0, g''(0) = -5, g'''(0) = 0, g^{(4)}(0) = 120$

3. The point is a local maximum.

4. Limit is $\frac{5}{9}$.

Rigorous Step-by-Step Derivations: Because the inner functions are even, we only compute x^2 and x^4 terms.

$$\cos(2x) = 1 - \frac{(2x)^2}{2} + \frac{(2x)^4}{24} = 1 - 2x^2 + \frac{2}{3}x^4 + o(x^4)$$

$$\cosh(4x) = 1 + \frac{(4x)^2}{2} + \frac{(4x)^4}{24} = 1 + 8x^2 + \frac{32}{3}x^4 + o(x^4)$$

Step 1: Division $Q(x) = \frac{\cos(2x)}{\cosh(4x)}$

$$Q(x) = \left(1 - 2x^2 + \frac{2}{3}x^4\right) \left(1 + 8x^2 + \frac{32}{3}x^4\right)^{-1}$$

Let $u = 8x^2 + \frac{32}{3}x^4$. Expansion of $(1 + u)^{-1} = 1 - u + u^2 + o(u^2)$.

$$u^2 = \left(8x^2 + \frac{32}{3}x^4\right)^2 = 64x^4 + o(x^4)$$

$$(1 + u)^{-1} = 1 - \left(8x^2 + \frac{32}{3}x^4\right) + 64x^4 = 1 - 8x^2 + \left(-\frac{32}{3} + \frac{192}{3}\right)x^4 = 1 - 8x^2 + \frac{160}{3}x^4$$

Multiply by the numerator:

$$\begin{aligned} Q(x) &= \left(1 - 2x^2 + \frac{2}{3}x^4\right) \left(1 - 8x^2 + \frac{160}{3}x^4\right) \\ &= 1(1 - 8x^2 + \frac{160}{3}x^4) - 2x^2(1 - 8x^2) + \frac{2}{3}x^4(1) + o(x^4) \\ &= 1 - 8x^2 + \frac{160}{3}x^4 - 2x^2 + 16x^4 + \frac{2}{3}x^4 \\ &= 1 - 10x^2 + x^4 \left(\frac{160}{3} + \frac{48}{3} + \frac{2}{3}\right) = 1 - 10x^2 + 70x^4 + o(x^4) \end{aligned}$$

Step 2: Square Root $R(x) = \sqrt{Q(x)}$ Let $v = -10x^2 + 70x^4$. Expansion of $(1 + v)^{1/2} = 1 + \frac{v}{2} - \frac{v^2}{8} + o(v^2)$.

$$\begin{aligned} v^2 &= (-10x^2 + 70x^4)^2 = 100x^4 + o(x^4) \\ R(x) &= 1 + \frac{1}{2}(-10x^2 + 70x^4) - \frac{1}{8}(100x^4) = 1 - 5x^2 + 35x^4 - \frac{25}{2}x^4 \\ &= 1 - 5x^2 + \left(\frac{70}{2} - \frac{25}{2}\right)x^4 = 1 - 5x^2 + \frac{45}{2}x^4 + o(x^4) \end{aligned}$$

Step 3: Outer Arctangent $g(x) = \arctan(R(x))$ We must recenter the expansion because $R(0) = 1$. Let $R(x) = 1 + w$ where $w = -5x^2 + \frac{45}{2}x^4$. The Taylor series of $\arctan(1 + w)$ at $w = 0$:

$$\begin{aligned} \arctan(1 + w) &= \arctan(1) + \frac{1}{1 + (1)^2}(w) + \frac{1}{2!} \left(\frac{-2(1)}{(1 + (1)^2)^2}\right) (w^2) + o(w^2) \\ &= \frac{\pi}{4} + \frac{w}{2} - \frac{w^2}{4} + o(w^2) \end{aligned}$$

Calculate $w^2 = (-5x^2 + \frac{45}{2}x^4)^2 = 25x^4 + o(x^4)$.

$$\begin{aligned} g(x) &= \frac{\pi}{4} + \frac{1}{2} \left(-5x^2 + \frac{45}{2}x^4\right) - \frac{1}{4}(25x^4) + o(x^4) \\ &= \frac{\pi}{4} - \frac{5}{2}x^2 + \frac{45}{4}x^4 - \frac{25}{4}x^4 = \frac{\pi}{4} - \frac{5}{2}x^2 + \frac{20}{4}x^4 \\ &= \frac{\pi}{4} - \frac{5}{2}x^2 + 5x^4 + o(x^4) \end{aligned}$$

Step 4: Derivatives and Limit Equating coefficients $g(x) = g(0) + \frac{g''(0)}{2}x^2 + \frac{g^{(4)}(0)}{24}x^4$:

$$\frac{g''(0)}{2} = -\frac{5}{2} \implies g''(0) = -5, \quad \frac{g^{(4)}(0)}{24} = 5 \implies g^{(4)}(0) = 120$$

Since $g''(0) < 0$, the function is concave down at 0, making it a local maximum. For the limit, expand the denominator:

$$1 - \cosh(3x) = 1 - \left(1 + \frac{(3x)^2}{2} + o(x^2)\right) = -\frac{9}{2}x^2 + o(x^2)$$

$$\lim_{x \rightarrow 0} \frac{\left(\frac{\pi}{4} - \frac{5}{2}x^2\right) - \frac{\pi}{4}}{-\frac{9}{2}x^2} = \lim_{x \rightarrow 0} \frac{-\frac{5}{2}x^2}{-\frac{9}{2}x^2} = \frac{5}{9}$$

Exercise 1.3:

Answers:

1. $h(x) = x + 1 + \frac{3}{2x} + \frac{7}{6x^2} + o\left(\frac{1}{x^2}\right)$

2. $(D) : y = x + 1$

3. The curve (C_h) is above (D) as $x \rightarrow +\infty$.

Rigorous Step-by-Step Derivations: Step 1: Change of Variable Let $t = \frac{1}{x}$. As $x \rightarrow +\infty$, $t \rightarrow 0^+$. Substitute into the function:

$$h\left(\frac{1}{t}\right) = \frac{\left(\frac{1}{t}\right)^2 + 1}{\frac{1}{t}} e^t = \left(\frac{1}{t} + t\right) e^t$$

Step 2: Limited Development of e^t We expand e^t to order 3 to ensure an order 2 remainder after multiplying by $\frac{1}{t}$:

$$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + o(t^3)$$

Step 3: Multiplication

$$\begin{aligned} h\left(\frac{1}{t}\right) &= \left(\frac{1}{t} + t\right) \left(1 + t + \frac{t^2}{2} + \frac{t^3}{6} + o(t^3)\right) \\ &= \frac{1}{t} \left(1 + t + \frac{t^2}{2} + \frac{t^3}{6} + o(t^3)\right) + t(1 + t + o(t)) \\ &= \frac{1}{t} + 1 + \frac{t}{2} + \frac{t^2}{6} + t + t^2 + o(t^2) \\ &= \frac{1}{t} + 1 + \frac{3}{2}t + \frac{7}{6}t^2 + o(t^2) \end{aligned}$$

Step 4: Reverting the Variable and Deductions Substitute $x = \frac{1}{t}$ back:

$$h(x) = x + 1 + \frac{3}{2x} + \frac{7}{6x^2} + o\left(\frac{1}{x^2}\right)$$

The slant asymptote (D) corresponds to the polynomial part as $x \rightarrow +\infty$:

$$y = x + 1$$

To find the position, examine the sign of the difference:

$$h(x) - y = \frac{3}{2x} + \frac{7}{6x^2} + o\left(\frac{1}{x^2}\right) \sim \frac{3}{2x}$$

Since $x \rightarrow +\infty$, $\frac{3}{2x} > 0$, meaning the curve is above the asymptote.

Exercise 1.4:

Answers:

1. Numerator main part: $\frac{x^2}{2}$. Denominator main part: $\frac{x^4}{2}$.
2. $\lim_{x \rightarrow 0} k(x) = +\infty$

Rigorous Step-by-Step Derivations: Step 1: Numerator Expansion Using the standard expansion of $\ln(1+x)$ up to order 2:

$$x - \ln(1+x) = x - \left(x - \frac{x^2}{2} + o(x^2) \right) = \frac{x^2}{2} + o(x^2)$$

The valuation (lowest degree) is $p = 2$.

Step 2: Denominator Expansion Using the expansion of $\cos(u)$ with $u = x^2$ up to order 2 in u (order 4 in x):

$$1 - \cos(x^2) = 1 - \left(1 - \frac{(x^2)^2}{2} + o(x^4) \right) = \frac{x^4}{2} + o(x^4)$$

The valuation is $q = 4$.

Step 3: Limit Evaluation Substitute the dominant parts into the limit:

$$k(x) = \frac{\frac{x^2}{2} + o(x^2)}{\frac{x^4}{2} + o(x^4)} = \frac{x^2 \frac{1}{2} + o(1)}{x^4 \frac{1}{2} + o(1)} \sim \frac{1}{x^2}$$

Because $p < q$ (numerator degree strictly less than denominator degree), the limit approaches infinity:

$$\lim_{x \rightarrow 0} k(x) = \lim_{x \rightarrow 0} \frac{1}{x^2} = +\infty$$

Part II:

Exercise 2.1:

Proof via Function Study: Let $h(x) = \cos(x) - 1 + \frac{x^2}{2}$. We want to prove $h(x) \geq 0$. Taking the derivative: $h'(x) = -\sin(x) + x$. From standard geometric trigonometric proofs, we know that $x \geq \sin x$ for all $x \geq 0$. Therefore, $h'(x) \geq 0$ on $[0, +\infty[$. Thus, h is strictly increasing on $[0, +\infty[$. Since $h(0) = \cos(0) - 1 + 0 = 0$, it follows that $h(x) \geq 0$ for all $x \geq 0$. Because both $\cos(x)$ and x^2 are even functions, $h(-x) = h(x)$. Therefore, the inequality $h(x) \geq 0$ holds symmetrically for $x < 0$, meaning $\cos(x) \geq 1 - \frac{x^2}{2}$ for all $x \in \mathbb{R}$.

Exercise 2.2:

Proof via Taylor-Lagrange: Apply the Taylor-Lagrange theorem to $f(x) = \sinh(x)$ at order 2 on the interval $[0, x]$.

$$f(0) = \sinh(0) = 0, \quad f'(0) = \cosh(0) = 1, \quad f''(0) = \sinh(0) = 0$$

The theorem states there exists a $c \in]0, x[$ such that:

$$\sinh(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(c)}{3!}x^3 = x + \frac{x^3}{6} \cosh(c)$$

We know the cosh function is strictly increasing on $[0, +\infty[$. Because $0 < c < x$, we can bound $\cosh(c)$:

$$\cosh(0) < \cosh(c) < \cosh(x) \implies 1 < \cosh(c) < \cosh(x)$$

Multiply the entire inequality by $\frac{x^3}{6}$ (which preserves the inequality direction since $x > 0$):

$$\frac{x^3}{6} < \frac{x^3}{6} \cosh(c) < \frac{x^3}{6} \cosh(x)$$

Add x to all sides to reconstruct our Taylor-Lagrange expression for $\sinh(x)$:

$$x + \frac{x^3}{6} < \sinh(x) < x + \frac{x^3}{6} \cosh(x)$$

Exercise 2.3:

Rigorous Step-by-Step Derivations: Expand both functions to order 4 to find the first non-canceling dominant term of their difference.

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)$$

For the right side, first expand the fraction:

$$\frac{\sin x}{x} = \frac{x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{120} + o(x^4)$$

Let $u = -\frac{x^2}{6} + \frac{x^4}{120}$. We cube the expression using $(1 + u)^3 = 1 + 3u + 3u^2 + u^3$.

$$\begin{aligned} u^2 &= \left(-\frac{x^2}{6} + \frac{x^4}{120}\right)^2 = \frac{x^4}{36} + o(x^4) \\ \left(\frac{\sin x}{x}\right)^3 &= 1 + 3\left(-\frac{x^2}{6} + \frac{x^4}{120}\right) + 3\left(\frac{x^4}{36}\right) + o(x^4) \\ &= 1 - \frac{x^2}{2} + \frac{x^4}{40} + \frac{x^4}{12} + o(x^4) \\ &= 1 - \frac{x^2}{2} + \left(\frac{3}{120} + \frac{10}{120}\right)x^4 = 1 - \frac{x^2}{2} + \frac{13}{120}x^4 + o(x^4) \end{aligned}$$

Calculate the difference $D(x) = \left(\frac{\sin x}{x}\right)^3 - \cos x$:

$$D(x) = \left(1 - \frac{x^2}{2} + \frac{13}{120}x^4\right) - \left(1 - \frac{x^2}{2} + \frac{5}{120}x^4\right) + o(x^4) = \frac{8}{120}x^4 + o(x^4) = \frac{x^4}{15} + o(x^4)$$

Because the lowest order remaining term $\frac{x^4}{15}$ is strictly positive for $x \neq 0$, the difference $D(x)$ is positive in a small neighborhood $]0, \alpha[$. Thus, $\cos(x) < \left(\frac{\sin x}{x}\right)^3$.

Part III:

Exercise 3.1:

Answer: $h(x) = -\frac{1}{12}x^{1/2} + \frac{7}{360}x^{3/2} + o(x^{3/2})$

Rigorous Step-by-Step Derivations: Because the denominator is $x^{3/2}$, we must expand the numerator functions to order 3 in terms of x to obtain a meaningful limit after division.

$$\begin{aligned} \cos(\sqrt{x}) &= 1 - \frac{(\sqrt{x})^2}{2!} + \frac{(\sqrt{x})^4}{4!} - \frac{(\sqrt{x})^6}{6!} + o(x^3) \\ &= 1 - \frac{x}{2} + \frac{x^2}{24} - \frac{x^3}{720} + o(x^3) \end{aligned}$$

$$\begin{aligned}
e^{-x/2} &= 1 + \left(-\frac{x}{2}\right) + \frac{1}{2!} \left(-\frac{x}{2}\right)^2 + \frac{1}{3!} \left(-\frac{x}{2}\right)^3 + o(x^3) \\
&= 1 - \frac{x}{2} + \frac{x^2}{8} - \frac{x^3}{48} + o(x^3)
\end{aligned}$$

Subtract the two expansions explicitly finding common denominators:

$$\begin{aligned}
\cos(\sqrt{x}) - e^{-x/2} &= \left(\frac{1}{24} - \frac{1}{8}\right)x^2 + \left(-\frac{1}{720} - \left(-\frac{1}{48}\right)\right)x^3 + o(x^3) \\
&= \left(\frac{1}{24} - \frac{3}{24}\right)x^2 + \left(-\frac{1}{720} + \frac{15}{720}\right)x^3 \\
&= -\frac{2}{24}x^2 + \frac{14}{720}x^3 = -\frac{1}{12}x^2 + \frac{7}{360}x^3 + o(x^3)
\end{aligned}$$

Divide term-by-term by $x^{3/2}$:

$$h(x) = \frac{-\frac{1}{12}x^2 + \frac{7}{360}x^3 + o(x^3)}{x^{3/2}} = -\frac{1}{12}x^{2-1.5} + \frac{7}{360}x^{3-1.5} + o(x^{3-1.5}) = -\frac{1}{12}x^{1/2} + \frac{7}{360}x^{3/2} + o(x^{3/2})$$

Exercise 3.2:

Answers:

1. $f'(x) = (1 + x^2)^{-1/2}$
2. $f'(x) = 1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 + o(x^4)$
3. $f(x) = x - \frac{x^3}{6} + \frac{3x^5}{40} + o(x^5)$

Rigorous Step-by-Step Derivations: Let $f(x) = \ln(x + \sqrt{1 + x^2})$. Differentiating using the chain rule $\frac{d}{dx} \ln(u) = \frac{u'}{u}$:

$$\begin{aligned}
f'(x) &= \frac{1}{x + \sqrt{1 + x^2}} \cdot \frac{d}{dx} \left(x + (1 + x^2)^{1/2}\right) \\
&= \frac{1}{x + \sqrt{1 + x^2}} \cdot \left(1 + \frac{1}{2}(1 + x^2)^{-1/2}(2x)\right) \\
&= \frac{1}{x + \sqrt{1 + x^2}} \cdot \left(1 + \frac{x}{\sqrt{1 + x^2}}\right) \\
&= \frac{1}{x + \sqrt{1 + x^2}} \cdot \left(\frac{\sqrt{1 + x^2} + x}{\sqrt{1 + x^2}}\right) = \frac{1}{\sqrt{1 + x^2}} = (1 + x^2)^{-1/2}
\end{aligned}$$

Let $u = x^2$. We expand $(1 + u)^{-1/2}$ using the generalized binomial theorem $(1 + u)^\alpha = 1 + \alpha u + \frac{\alpha(\alpha-1)}{2}u^2$:

$$\begin{aligned}
(1 + u)^{-1/2} &= 1 + \left(-\frac{1}{2}\right)u + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}u^2 + o(u^2) \\
&= 1 - \frac{1}{2}u + \frac{3}{8}u^2 + o(u^2)
\end{aligned}$$

Substituting $u = x^2$:

$$f'(x) = 1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 + o(x^4)$$

Integrate the polynomial term by term to find $f(x)$. The constant of integration $C = f(0) = \ln(0 + \sqrt{1}) = 0$.

$$f(x) = \int_0^x \left(1 - \frac{1}{2}t^2 + \frac{3}{8}t^4\right) dt = x - \frac{1}{2} \left(\frac{x^3}{3}\right) + \frac{3}{8} \left(\frac{x^5}{5}\right) + o(x^5) = x - \frac{x^3}{6} + \frac{3x^5}{40} + o(x^5)$$

Part IV:

Exercise 4.1:

Answer: $\omega(\theta) = \theta + \frac{7}{12}\theta^3 + \frac{109}{480}\theta^5 + o(\theta^5)$

Rigorous Step-by-Step Derivations:

$$\omega(\theta) = \tan \theta \cdot \left(1 - \frac{1}{2} \sin^2 \theta\right)^{-1/2}$$

First, determine the expansion of $-\frac{1}{2} \sin^2 \theta$. We need $\sin \theta$ to order 4 to square it correctly.

$$\sin \theta = \theta - \frac{\theta^3}{6} + o(\theta^4)$$

$$\begin{aligned}\sin^2 \theta &= \left(\theta - \frac{\theta^3}{6}\right)^2 = \theta^2 - 2(\theta)\left(\frac{\theta^3}{6}\right) + o(\theta^5) = \theta^2 - \frac{\theta^4}{3} + o(\theta^5) \\ -\frac{1}{2} \sin^2 \theta &= -\frac{1}{2}\theta^2 + \frac{1}{6}\theta^4 + o(\theta^5)\end{aligned}$$

Let $u = -\frac{1}{2}\theta^2 + \frac{1}{6}\theta^4$. Expand $(1+u)^{-1/2} = 1 - \frac{1}{2}u + \frac{3}{8}u^2 + o(u^2)$.

$$u^2 = \left(-\frac{1}{2}\theta^2 + \frac{1}{6}\theta^4\right)^2 = \frac{1}{4}\theta^4 + o(\theta^4)$$

$$\begin{aligned}(1+u)^{-1/2} &= 1 - \frac{1}{2}\left(-\frac{1}{2}\theta^2 + \frac{1}{6}\theta^4\right) + \frac{3}{8}\left(\frac{1}{4}\theta^4\right) + o(\theta^4) \\ &= 1 + \frac{1}{4}\theta^2 - \frac{1}{12}\theta^4 + \frac{3}{32}\theta^4 \\ &= 1 + \frac{1}{4}\theta^2 + \left(-\frac{8}{96} + \frac{9}{96}\right)\theta^4 = 1 + \frac{1}{4}\theta^2 + \frac{1}{96}\theta^4 + o(\theta^4)\end{aligned}$$

Multiply by the expansion of $\tan \theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + o(\theta^5)$:

$$\begin{aligned}\omega(\theta) &= \left(\theta + \frac{1}{3}\theta^3 + \frac{2}{15}\theta^5\right) \left(1 + \frac{1}{4}\theta^2 + \frac{1}{96}\theta^4\right) \\ &= \theta \left(1 + \frac{1}{4}\theta^2 + \frac{1}{96}\theta^4\right) + \frac{1}{3}\theta^3 \left(1 + \frac{1}{4}\theta^2\right) + \frac{2}{15}\theta^5(1) + o(\theta^5) \\ &= \theta + \frac{1}{4}\theta^3 + \frac{1}{96}\theta^5 + \frac{1}{3}\theta^3 + \frac{1}{12}\theta^5 + \frac{2}{15}\theta^5 \\ &= \theta + \theta^3 \left(\frac{1}{4} + \frac{1}{3}\right) + \theta^5 \left(\frac{1}{96} + \frac{1}{12} + \frac{2}{15}\right)\end{aligned}$$

Solve the common denominators:

$$\begin{aligned}\theta^3 : \frac{3}{12} + \frac{4}{12} &= \frac{7}{12} \\ \theta^5 : \frac{1 \cdot 5}{96 \cdot 5} + \frac{1 \cdot 40}{12 \cdot 40} + \frac{2 \cdot 32}{15 \cdot 32} &= \frac{5}{480} + \frac{40}{480} + \frac{64}{480} = \frac{109}{480}\end{aligned}$$

Final result:

$$\omega(\theta) = \theta + \frac{7}{12}\theta^3 + \frac{109}{480}\theta^5 + o(\theta^5)$$