Set Correction with notes

Category 1:

- I_1**:** Look closely: $(e^x \cos x)' = e^x \cos x + e^x(-\sin x) = e^x(\cos x - \sin x)$.

$$I_1 = \Big[e^x \cos x\Big]_0^{\pi/2} = (e^{\pi/2}(0)) - (e^0(1)) = -\mathbf{1}$$

- I_2**:** Notice the x^2 in the denominator. This screams quotient rule. Test $(\frac{\sin x}{x})'$:

$$\left(\frac{\sin x}{x}\right)' = \frac{x(\cos x) - (\sin x)(1)}{x^2}$$

$$I_2 = \left[\frac{\sin x}{x}\right]_1^{\pi} = \frac{\sin \pi}{\pi} - \frac{\sin 1}{1} = -\sin(\mathbf{1})$$

- I_3**:** Expand the fraction: $\frac{xe^x - e^x}{x^2}$. This is exactly $(\frac{e^x}{x})'$.

$$I_3 = \left[\frac{e^x}{x}\right]_1^2 = \frac{\mathbf{e^2}}{\mathbf{2}} - \mathbf{e}$$

Category 2:

- I_4**: Trick:** The numerator is 1. Add and subtract x^4: $1 = 1 + x^4 - x^4$.

$$\frac{1 + x^4 - x^4}{x(1 + x^4)} = \frac{1 + x^4}{x(1 + x^4)} - \frac{x^4}{x(1 + x^4)} = \frac{1}{x} - \frac{x^3}{1 + x^4}$$

  Now both are standard primitives ($\frac{1}{x}$ and $\frac{1}{4}\frac{u'}{u}$).

$$I_4 = \left[\ln|x| - \frac{1}{4}\ln(1 + x^4)\right]_1^2 = \frac{\mathbf{5}}{\mathbf{4}}\ln \mathbf{2} - \frac{\mathbf{1}}{\mathbf{4}}\ln \mathbf{17}$$

- I_5**: Trick:** Multiply top and bottom by e^x.

$$\int \frac{e^x}{e^{2x} + 1}\,dx$$

  This perfectly matches $\frac{u'}{u^2+1}$ where $u = e^x$.

$$I_5 = \Big[\arctan(e^x)\Big]_0^{\ln 3} = \arctan(3) - \arctan(1) = \arctan(\mathbf{3}) - \frac{\pi}{\mathbf{4}}$$

- I_6**: Trick:** Use the definitions: $\cosh x + \sinh x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = e^x$. The integral is simply $\int_0^1 e^{-x} dx$.

$$I_6 = \Big[-e^{-x}\Big]_0^1 = \mathbf{1} - \mathbf{e^{-1}}$$

Category 3:

- I_7**:** Notice that the numerator x is related to the derivative of x^2. Manipulate the denominator to be a quadratic in terms of x^2.

$$x^4+x^2+1=(x^2)^2+x^2+1=\left(x^2+\frac{1}{2}\right)^2+\frac{3}{4}=\frac{3}{4}\left[\left(\frac{2x^2+1}{\sqrt{3}}\right)^2+1\right]$$

The derivative of $u(x)=\frac{2x^2+1}{\sqrt{3}}$ is $u'(x)=\frac{4x}{\sqrt{3}}$. Adjust constants to form $\frac{u'}{u^2+1}$.

$$I_7=\frac{1}{2\sqrt{3}}\Big[\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)\Big]_0^1=\frac{\pi}{\mathbf{12\sqrt{3}}}$$

- I_8**:** Complete the square directly in the denominator since $\Delta<0$:

$$x^2+x+1=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}=\frac{3}{4}\left[\left(\frac{2x+1}{\sqrt{3}}\right)^2+1\right]$$

The integral is clearly related to arctangent.

$$I_8=\frac{2}{\sqrt{3}}\Big[\arctan\left(\frac{2x+1}{\sqrt{3}}\right)\Big]_{-1/2}^0=\frac{2}{\sqrt{3}}\left(\arctan\left(\frac{1}{\sqrt{3}}\right)-\arctan(0)\right)=\frac{\pi}{\mathbf{3\sqrt{3}}}$$

Category 4:

- I_9**:** Let $f(x)=\ln(x+\sqrt{x^2+1})$.

$$f(-x)=\ln(-x+\sqrt{x^2+1})=\ln\left(\frac{x^2+1-x^2}{x+\sqrt{x^2+1}}\right)=\ln\left(\frac{1}{x+\sqrt{x^2+1}}\right)=-f(x)$$

The function is strictly odd on a symmetric interval. $\mathbf{I_9=0}$.

- I_{10}: $f(-x)=\frac{(-x)^4\sin(-x^3)+(-x)^3\cos(-x)}{1+(-x)^6}=\frac{-x^4\sin(x^3)-x^3\cos(x)}{1+x^6}=-f(x)$. Odd function on a symmetric interval. $\mathbf{I_{10}=0}$.

Category 5:

- I_{11}**: Trick:** Add and subtract 1 to force the derivative of tangent.

$$\tan^2(x)=(\tan^2(x)+1)-1$$

Both are now standard table primitives ($u'\to u$).

$$I_{11}=\Big[\tan x-x\Big]_0^{\pi/4}=(\tan(\pi/4)-\pi/4)-(0)=\mathbf{1}-\frac{\pi}{\mathbf{4}}$$

- I_{12}**: Trick:** Expand $\cos(2x)$ as $\cos^2 x-\sin^2 x$. The fraction immediately separates:

$$\frac{\cos^2 x-\sin^2 x}{\sin^2 x\cos^2 x}=\frac{1}{\sin^2 x}-\frac{1}{\cos^2 x}$$

$$I_{12}=\Big[-1/\tan x-\tan x\Big]_{\pi/4}^{\pi/3}=\left(-\frac{\sqrt{3}}{3}-\sqrt{3}\right)-(-1-1)=\mathbf{2}-\frac{\mathbf{4\sqrt{3}}}{\mathbf{3}}$$

- I_{13}**: Trick:** Use the product-to-sum trigonometric formula: $\sin a\cos b = \frac{1}{2}(\sin(a+b)+\sin(a-b))$.
$$\sin(3x)\cos(2x) = \frac{1}{2}(\sin(5x)+\sin(x))$$
$$I_{13} = \frac{1}{2}\Big[-\frac{1}{5}\cos(5x)-\cos(x)\Big]_0^{\pi/2} = \frac{1}{2}\left((0-0)-\left(-\frac{1}{5}-1\right)\right) = \frac{1}{2}\left(\frac{6}{5}\right) = \mathbf{\frac{3}{5}}$$

- I_{14}**: Trick:** Recognize the perfect square inside the root by substituting $1 = \sin^2 x+\cos^2 x$.
$$\sqrt{1+\sin(2x)} = \sqrt{\sin^2 x+\cos^2 x+2\sin x\cos x} = \sqrt{(\sin x+\cos x)^2}$$
Since $x \in [0,\pi/4]$, $\sin x+\cos x > 0$, so we can drop the absolute value.
$$I_{14} = \int_0^{\pi/4} \frac{\sin x+\cos x}{\sin x+\cos x}dx = \int_0^{\pi/4} 1\,dx = \frac{\pi}{\mathbf{4}}$$

- I_{15}**: Trick:** Multiply by the conjugate $(1-\sin x)$.
$$\frac{1-\sin x}{1-\sin^2 x} = \frac{1-\sin x}{\cos^2 x} = \frac{1}{\cos^2 x}-\frac{\sin x}{\cos^2 x}$$
$$I_{15} = \Big[\tan x-\frac{1}{\cos x}\Big]_0^{\pi/2} = \lim_{x\to\pi/2}\left(\frac{\sin x-1}{\cos x}\right)-(-1)$$
By L'Hôpital's rule at $\pi/2$, the limit is 0. Thus, $I_{15} = 0-(-1) = \mathbf{1}$.

- I_{16}**:** We know that:
$$(\sin^2 x+\cos^2 x)^2 = \sin^4 x+2\sin^2 x\cos^2 x+\cos^4 x$$
Since $\sin^2 x+\cos^2 x = 1$, we can plug that in $(1^2 = 1)$ and isolate the terms we need:
$$1 = \sin^4 x+\cos^4 x+2\sin^2 x\cos^2 x \implies \sin^4 x+\cos^4 x = 1-2\sin^2 x\cos^2 x$$
Using $\sin(2x) = 2\sin x\cos x$, squaring it gives $\sin^2(2x) = 4\sin^2 x\cos^2 x$. Therefore, exactly half of that is:
$$2\sin^2 x\cos^2 x = \frac{1}{2}\sin^2(2x)$$
Substitute this back in:
$$\sin^4 x+\cos^4 x = 1-\frac{1}{2}\sin^2(2x)$$
To actually integrate this, we need to eliminate the squared sine term using the power-reduction formula $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$. Here, our angle is $\theta = 2x$, so it doubles to $4x$:
$$\sin^2(2x) = \frac{1-\cos(4x)}{2}$$
Substitute this into our expression:
$$1-\frac{1}{2}\left(\frac{1-\cos(4x)}{2}\right) = 1-\frac{1-\cos(4x)}{4}$$
Distribute the constants and simplify:
$$= 1-\frac{1}{4}+\frac{1}{4}\cos(4x) = \frac{3}{4}+\frac{1}{4}\cos(4x)$$
Now that we have reduced the expression into simple, linear trigonometric terms, the integration is incredibly straightforward:
$$\int(\sin^4 x+\cos^4 x)\,dx = \int\left(\frac{3}{4}+\frac{1}{4}\cos(4x)\right)\,dx$$
Integrating term by term gives the final answer:
$$= \mathbf{\frac{3}{4}x}+\mathbf{\frac{1}{16}}\sin(\mathbf{4x})+\mathbf{C}$$

Category 6:

Note: In general, look for the $1/n$ factor outside, and k/n inside. If it's not there, use ln(), Taylor-Young (LD), or shift the indices, and always don't forget to mention that the sum is continuous.

- S_1**:** Take the natural log of the product to turn it into a sum:
$$\ln(P_n) = \sum_{k=1}^{n} \frac{k}{n^2+k^2} = \frac{1}{n}\sum_{k=1}^{n} \frac{k/n}{1+(k/n)^2}$$
This is a Riemann sum on $[0,1]$ for $f(x) = \frac{x}{1+x^2}$.
$$\lim \ln(P_n) = \int_0^1 \frac{x}{1+x^2}dx = \left[\frac{1}{2}\ln(1+x^2)\right]_0^1 = \frac{1}{2}\ln 2 = \ln(\sqrt{2})$$
Therefore, $\mathbf{S_1 = \sqrt{2}}$.

- S_2**:** Factor out n^2 from the denominator to create the $1/n$ step:
$$S_2 = \sum_{k=1}^{n} \frac{n\arctan(k/n)}{n^2(1+(k/n)^2)} = \frac{1}{n}\sum_{k=1}^{n} \frac{\arctan(k/n)}{1+(k/n)^2}$$
This is a Riemann sum for $f(x) = \frac{\arctan(x)}{1+x^2}$. Notice this is exactly $u'u^1$.
$$\lim S_2 = \int_0^1 \frac{\arctan x}{1+x^2}dx = \left[\frac{1}{2}(\arctan x)^2\right]_0^1 = \frac{1}{2}\left(\frac{\pi}{4}\right)^2 = \boldsymbol{\frac{\pi^2}{32}}$$

- S_3**:** Use the Limited Development $\cos(u) = 1 - u^2/2 + o(u^2)$ as $u \to 0$. Here, $u = \frac{1}{\sqrt{n^2+nk}}$, which goes to 0 as $n \to \infty$.
$$1 - \cos\left(\frac{1}{\sqrt{n^2+nk}}\right) \approx \frac{1}{2(n^2+nk)}$$
Substitute this into the sequence:
$$S_3 = \sum_{k=1}^{n} n\left[\frac{1}{2(n^2+nk)}\right] = \sum_{k=1}^{n} \frac{n}{2n^2(1+k/n)} = \frac{1}{n}\sum_{k=1}^{n} \frac{1}{2(1+k/n)}$$
This is a Riemann sum for $f(x) = \frac{1}{2(1+x)}$ on $[0,1]$.
$$\lim S_3 = \int_0^1 \frac{1}{2(1+x)}dx = \left[\frac{1}{2}\ln(1+x)\right]_0^1 = \boldsymbol{\frac{1}{2}\ln 2}$$

- S_4**:** The sum bounds are $n+1$ to $2n$. Let's shift the index: let $j = k-n$. When $k = n+1$, $j = 1$. When $k = 2n$, $j = n$. So we sum from $j = 1$ to n. Substitute $k = n+j$ into the denominator:
$$k^2 - nk + n^2 = (n+j)^2 - n(n+j) + n^2 = (n^2+2nj+j^2) - n^2 - nj + n^2 = n^2 + nj + j^2$$
So the sum becomes:
$$S_4 = \sum_{j=1}^{n} \frac{n}{n^2+nj+j^2} = \frac{1}{n}\sum_{j=1}^{n} \frac{1}{1+(j/n)+(j/n)^2}$$
This is a Riemann sum for $f(x) = \frac{1}{x^2+x+1}$ on $[0,1]$. Complete the square: $x^2+x+1 = (x+1/2)^2 + 3/4 = \frac{3}{4}\left[\left(\frac{2x+1}{\sqrt{3}}\right)^2 + 1\right]$.
$$\lim S_4 = \frac{2}{\sqrt{3}}\left[\arctan\left(\frac{2x+1}{\sqrt{3}}\right)\right]_0^1 = \frac{2}{\sqrt{3}}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \boldsymbol{\frac{\pi}{3\sqrt{3}}}$$