# Integrals – Detailed Correction

*Focus on Method Justification, Substitutions, and Structural Synthesis*

---

**How to use this document:** Each solution follows a fixed structure: **Method** (scan for technique) → **Why** (understand the idea) → **Execution** (step-by-step algebra) → **Pitfall** (what students commonly get wrong). Difficulty flags: **[Standard]** — can be done quickly under exam pressure. **[Tricky]** — requires a non-obvious first step. **[Time-Sink]** — correct but slow; budget your time.

## Category 1: Integration by Parts (IBP)

1. $I_1 = \displaystyle\int x\tan^2(x)\,dx$ **[Standard]**

***Method:*** *Trigonometric reduction, then IBP.*

***Why we do this:***

$\tan^2 x$ is not directly integrable in the way polynomials are. Rewriting it as $\sec^2 x - 1$ splits the integral into one part that is the perfect IBP target ($x\sec^2 x$, whose antiderivative of $\sec^2 x$ is known) and one trivial polynomial part.

**Execution:** Use $\tan^2 x = \sec^2 x - 1$:

$$I_1 = \int x\sec^2 x\,dx - \int x\,dx.$$

IBP on the first term with $u = x$, $\boldsymbol{dv = \sec^2 x\,dx}$, so $du = dx$, $\boldsymbol{v = \tan x}$:

$$\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx = x\tan x + \ln|\cos x| + C.$$

$$I_1 = x\tan x + \ln|\cos x| - \frac{x^2}{2} + C.$$

$$\boxed{I_1 = x\tan x + \ln|\cos x| - \frac{x^2}{2} + C}$$

**▶ Common Pitfall:**

Students often forget the sign: $\int \tan x\,dx = -\ln|\cos x| + C$, not $+\ln|\cos x|$. Write $-\int \tan x\,dx = +\ln|\cos x|$ explicitly every time.

2. $I_2 = \displaystyle\int \cos(\ln x)\,dx$ **[Tricky]**

***Method:*** *Hidden cyclic IBP applied twice.*

***Why we do this:***

When IBP produces an integral that loops back to the original, the two expressions form an algebraic equation. Instead of integrating forever, collect the two occurrences of $I_2$ on the same side and divide. This "solve for the integral" trick is the signature of cyclic IBP.

**Execution:**

**First IBP:** Set $u = \cos(\ln x)$, $\boldsymbol{dv = dx}$. Then $du = -\dfrac{\sin(\ln x)}{x}\,dx$, $\boldsymbol{v = x}$:

$$I_2 = x\cos(\ln x) + \int \sin(\ln x)\,dx.$$

**Second IBP** on $\displaystyle\int \sin(\ln x)\,dx$: Set $u = \sin(\ln x)$, $\boldsymbol{dv = dx}$. Then $du = \dfrac{\cos(\ln x)}{x}\,dx$, $\boldsymbol{v = x}$:

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - \int \cos(\ln x)\,dx = x\sin(\ln x) - I_2.$$

**Substitute back:**

$$I_2 = x\cos(\ln x) + x\sin(\ln x) - I_2 \implies 2I_2 = x\big(\cos(\ln x) + \sin(\ln x)\big).$$

$$\boxed{I_2 = \frac{x}{2}\big(\cos(\ln x) + \sin(\ln x)\big) + C}$$

**▶ Common Pitfall:**

After the second IBP, the $+C$ from the second integral is absorbed into the final $+C$. Do not write $2I_2 = \ldots + C$ and then carry two separate constants — there is only one arbitrary constant at the end.

3. $I_3 = \displaystyle\int_0^{\pi} e^{-x}\sin(x)\,dx$ **[Standard]**

***Method:*** *Cyclic IBP over a bounded interval.*

***Why we do this:***

The product $e^{-x}\sin x$ is a classic cyclic pair: neither function disappears under differentiation. Two IBP passes reproduce the original integral, allowing us to solve algebraically. On a definite integral, the boundary term must be evaluated before the algebraic step.

**Execution:**

**First IBP:** $u = \sin x$, $\boldsymbol{dv = e^{-x}\,dx} \Rightarrow du = \cos x\,dx$, $\boldsymbol{v = -e^{-x}}$:

$$I_3 = \Big[-e^{-x}\sin x\Big]_0^{\pi} + \int_0^{\pi} e^{-x}\cos x\,dx = 0 + \int_0^{\pi} e^{-x}\cos x\,dx.$$

**Second IBP** on $\displaystyle\int_0^{\pi} e^{-x}\cos x\,dx$: $u = \cos x$, $\boldsymbol{dv = e^{-x}\,dx} \Rightarrow du = -\sin x\,dx$, $\boldsymbol{v = -e^{-x}}$:

$$\int_0^{\pi} e^{-x}\cos x\,dx = \Big[-e^{-x}\cos x\Big]_0^{\pi} - \int_0^{\pi} e^{-x}\sin x\,dx = \big(-e^{-\pi}(-1) - (-1)(1)\big) - I_3 = (e^{-\pi} + 1) - I_3.$$

**Solve:**

$$I_3 = e^{-\pi} + 1 - I_3 \implies 2I_3 = e^{-\pi} + 1.$$

$$\boxed{I_3 = \frac{e^{-\pi}+1}{2}}$$

**▶ Common Pitfall:**

When evaluating $\left[-e^{-x}\cos x\right]_0^{\pi}$: at $x = \pi$, $\cos\pi = -1$ so the term is $-e^{-\pi}\cdot(-1) = e^{-\pi}$. At $x = 0$, $\cos 0 = 1$ so the term is $-e^{0}\cdot 1 = -1$. The bound contribution is $e^{-\pi} - (-1) = e^{-\pi} + 1$. Students frequently drop the negative sign on the lower bound.

4. $I_4 = \displaystyle\int \frac{x\ln\left(x+\sqrt{x^2+1}\right)}{\sqrt{x^2+1}}\,dx$ **[Tricky]**

***Method:*** *IBP using the known derivative of* $\operatorname{arcsinh} x = \ln(x+\sqrt{x^2+1})$.

***Why we do this:***

The integrand contains $\ln(x+\sqrt{x^2+1})$, which by LIATE should be $u$. The key is recognising that its derivative simplifies beautifully, and that $\frac{x}{\sqrt{x^2+1}}$ is the derivative of $\sqrt{x^2+1}$, making the choice of $v$ natural.

**Derivation of the key derivative (do not skip):**

$$\frac{d}{dx}\left[\ln\left(x+\sqrt{x^2+1}\right)\right] = \frac{1}{x+\sqrt{x^2+1}}\cdot\frac{d}{dx}\left[x+\sqrt{x^2+1}\right] = \frac{1}{x+\sqrt{x^2+1}}\cdot\left(1+\frac{x}{\sqrt{x^2+1}}\right).$$

Simplify the bracket: $1 + \dfrac{x}{\sqrt{x^2+1}} = \dfrac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}$. So:

$$\frac{d}{dx}\left[\ln(x+\sqrt{x^2+1})\right] = \frac{1}{x+\sqrt{x^2+1}}\cdot\frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}}.$$

**Execution:** Set $u = \ln(x+\sqrt{x^2+1})$, $\boldsymbol{dv = \dfrac{x}{\sqrt{x^2+1}}\,dx}$. Then $du = \dfrac{dx}{\sqrt{x^2+1}}$, $\boldsymbol{v = \sqrt{x^2+1}}$:

$$I_4 = \sqrt{x^2+1}\ln\left(x+\sqrt{x^2+1}\right) - \int\frac{\sqrt{x^2+1}}{\sqrt{x^2+1}}\,dx = \sqrt{x^2+1}\ln\left(x+\sqrt{x^2+1}\right) - \int 1\,dx.$$

$$\boxed{I_4 = \sqrt{x^2+1}\ln\left(x+\sqrt{x^2+1}\right) - x + C}$$

**▶ Common Pitfall:**

Do not use $\frac{d}{dx}[\ln(x+\sqrt{x^2+1})] = \frac{1}{\sqrt{x^2+1}}$ from memory without verification. The chain rule derivation above confirms it. Getting the derivative wrong here corrupts the entire IBP.

5. $I_5 = \displaystyle\int \frac{x^2e^x}{(x+2)^2}\,dx$ **[Tricky]**

***Method:*** *Force the numerator to match the denominator structure, then IBP.*

***Why we do this:***

The integrand does not have an obvious $u$ and $dv$ from LIATE. The trick is algebraic: rewrite $x^2$ in terms of $(x+2)$ so that one piece of the integral cancels with the IBP derivative term. This is the "telescope" technique for rational-exponential integrals.

**Execution:** Write $x^2 = (x-2)(x+2)+4$:

$$I_5 = \int \frac{(x-2)(x+2)}{(x+2)^2}e^x\,dx + \int \frac{4}{(x+2)^2}e^x\,dx = \int \frac{x-2}{x+2}e^x\,dx + \int \frac{4}{(x+2)^2}e^x\,dx.$$

Apply IBP to the first term with $u = \dfrac{x-2}{x+2}$, $\boldsymbol{dv = e^x\,dx}$, so $\boldsymbol{v = e^x}$ and

$$du = \frac{(x+2)-(x-2)}{(x+2)^2}\,dx = \frac{4}{(x+2)^2}\,dx.$$

$$\int \frac{x-2}{x+2}e^x\,dx = \frac{x-2}{x+2}e^x - \int \frac{4}{(x+2)^2}e^x\,dx.$$

The second integral cancels exactly:

$$I_5 = \frac{x-2}{x+2}e^x - \int \frac{4}{(x+2)^2}e^x\,dx + \int \frac{4}{(x+2)^2}e^x\,dx.$$

$$\boxed{I_5 = \frac{x-2}{x+2}\,e^x + C}$$

**► Common Pitfall:**

The cancellation only works if you compute $du$ correctly. $\dfrac{d}{dx}\left[\dfrac{x-2}{x+2}\right] = \dfrac{4}{(x+2)^2}$, not $\dfrac{1}{x+2}$. Use the quotient rule explicitly.

6. $I_6 = \displaystyle\int \frac{x^2}{(x\sin x+\cos x)^2}\,dx$ **[Time-Sink]**

***Method:*** *Advanced IBP: spot the hidden derivative, then split the integrand.*

***Why we do this:***

No standard technique applies directly. The key is noticing that $\frac{d}{dx}(x\sin x + \cos x) = x\cos x$, which appears in the numerator after a strategic split. This transforms the integral into a perfect IBP where $dv$ is the derivative of a quotient, making $v$ computable.

**Execution:**

**Observe:** $\dfrac{d}{dx}(x\sin x+\cos x) = \sin x + x\cos x - \sin x = x\cos x.$

**Rewrite:**

$$\frac{x^2}{(x\sin x+\cos x)^2} = \frac{x}{\cos x}\cdot\frac{x\cos x}{(x\sin x+\cos x)^2}.$$

The second factor is $\dfrac{d}{dx}\left[\dfrac{-1}{x\sin x+\cos x}\right]$.

Set $u = \dfrac{x}{\cos x}$, $\boldsymbol{dv = \dfrac{x\cos x}{(x\sin x + \cos x)^2}\,dx}$, so $\boldsymbol{v = \dfrac{-1}{x\sin x + \cos x}}$ and

$$du = \frac{\cos x + x\sin x}{\cos^2 x}\,dx.$$

$$I_6 = \frac{-x}{\cos x(x\sin x + \cos x)} + \int \frac{\cos x + x\sin x}{\cos^2 x(x\sin x + \cos x)}\,dx.$$

The remaining integral simplifies: $\dfrac{\cos x + x\sin x}{\cos^2 x(x\sin x + \cos x)} = \dfrac{1}{\cos^2 x} = \sec^2 x$, so $\displaystyle\int \sec^2 x\,dx =$ $\tan x$.

$$\boxed{I_6 = \frac{-x}{\cos x(x\sin x + \cos x)} + \tan x + C}$$

**▶ Common Pitfall:**

This integral is not solvable by naive IBP (picking $u = x^2$ etc.). If you attempt standard LIATE without the key observation above, you will loop forever. Always check if the denominator's derivative appears anywhere in the numerator before applying IBP.

## Category 2: Variable Change

7. $I_7 = \displaystyle\int \frac{1}{\sqrt{x} + \sqrt[3]{x}}\,dx$ **[Tricky]**

***Method:*** *LCM substitution to clear all fractional exponents simultaneously.*

***Why we do this:***

With two different roots ($\sqrt{x} = x^{1/2}$ and $\sqrt[3]{x} = x^{1/3}$), no single obvious substitution works. Setting $x = t^n$ where $n$ is the LCM of all denominators (here $\operatorname{lcm}(2,3) = 6$) converts every root to an integer power of $t$, reducing the problem to polynomial long division.

**Execution: Sub:** $\boldsymbol{x = t^6 \Rightarrow dx = 6t^5\,dt, \quad \sqrt{x} = t^3, \quad \sqrt[3]{x} = t^2.}$

$$I_7 = \int \frac{6t^5}{t^3 + t^2}\,dt = 6\int \frac{t^3}{t+1}\,dt.$$

Polynomial long division: $\dfrac{t^3}{t+1} = t^2 - t + 1 - \dfrac{1}{t+1}$.

$$I_7 = 6\int\left(t^2 - t + 1 - \frac{1}{t+1}\right)dt = 6\left(\frac{t^3}{3} - \frac{t^2}{2} + t - \ln|t+1|\right).$$

Back-substitute $t = x^{1/6}$:

$$\boxed{I_7 = 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ln\left(1 + \sqrt[6]{x}\right) + C}$$

**▶ Common Pitfall:**

Back-substitution is mandatory. After computing in $t$, every $t$ must become $x^{1/6}$ before writing the final answer. Leaving the answer in $t$ is an automatic deduction.

8. $I_8 = \displaystyle\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$ **[Tricky]**

***Method:*** *Hyperbolic substitution to clear $\sqrt{x^2+a^2}$.*

***Why we do this:***

The form $\sqrt{x^2+a^2}$ calls for $x = a\sinh t$ because the identity $\cosh^2 t - \sinh^2 t = 1$ gives $\sqrt{a^2\sinh^2 t + a^2} = a\cosh t$ with no square root left. This is cleaner than trigonometric substitution for this family.

**Execution:** **Sub:** $\boldsymbol{x = 2\sinh t \Rightarrow dx = 2\cosh t\,dt, \quad \sqrt{x^2+4} = 2\cosh t.}$

$$I_8 = \int \frac{2\cosh \mathbf{t\,dt}}{4\sinh^2 t\cdot 2\cosh t} = \frac{1}{4}\int \operatorname{csch}^2(t)\,dt = -\frac{1}{4}\coth t + C.$$

Back-substitute: $\coth t = \dfrac{\cosh t}{\sinh t} = \dfrac{\sqrt{x^2+4}/2}{x/2} = \dfrac{\sqrt{x^2+4}}{x}$.

$$\boxed{I_8 = -\frac{\sqrt{x^2+4}}{4x} + C}$$

**► Common Pitfall:**

When back-substituting, $\coth t \neq \tanh t$. Recompute $\cosh t$ and $\sinh t$ from $x = 2\sinh t$: $\sinh t = x/2$, $\cosh t = \sqrt{1+\sinh^2 t} = \sqrt{1+x^2/4} = \sqrt{x^2+4}/2$.

9. $I_9 = \displaystyle\int \frac{\sqrt{x}}{1+x^3}\,dx$ **[Tricky]**

***Method:*** *Power substitution linking $\sqrt{x}$ and $x^3$ simultaneously.*

***Why we do this:***

The numerator $\sqrt{x} = x^{1/2}$ and denominator $x^3$ share a common structure if we set $t = x^{3/2}$: then $x^3 = t^2$ (denominator becomes $1+t^2$) and the $\sqrt{x}\,dx$ in the numerator becomes $\frac{2}{3}\,dt$. The integral collapses to the standard arctan form.

**Execution:** **Sub:** $\boldsymbol{t = x^{3/2} \Rightarrow dt = \tfrac{3}{2}\sqrt{x}\,dx \Rightarrow \sqrt{x}\,dx = \tfrac{2}{3}\,dt. \quad x^3 = t^2.}$

$$I_9 = \int \frac{\sqrt{x}\,dx}{1+x^3} = \frac{2}{3}\int \frac{dt}{1+t^2} = \frac{2}{3}\arctan(t).$$

$$\boxed{I_9 = \frac{2}{3}\arctan\left(x^{3/2}\right) + C}$$

**► Common Pitfall:**

The substitution gives $\sqrt{x}\,dx = \frac{2}{3}\,dt$, *not* $dx = \frac{2}{3}\,dt$. The $\sqrt{x}$ in the numerator is consumed by the differential. Missing this produces a wrong integrand.

10. $I_{10} = \displaystyle\int \frac{\cos 2x}{\sin^4 x + \cos^4 x}\,dx$ **[Tricky]**

***Method:*** *Trig identity to simplify the denominator, then u-substitution.*

***Why we do this:***

The denominator $\sin^4 x + \cos^4 x$ looks complex but factors via the algebraic identity $(a^2 + b^2)^2 = (a^2 + b^2)^2 - 2a^2b^2$ (with $a = \sin x$, $b = \cos x$). This reveals $\sin^2(2x)$ in the denominator, which pairs naturally with $\cos 2x = \frac{1}{2} \cdot \frac{d}{dx}[\sin 2x]$ in the numerator.

**Execution:**

**Step 1: Simplify the denominator.**

$$\sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x = 1 - \tfrac{1}{2}\sin^2(2x).$$

$$I_{10} = \int \frac{\cos 2x}{1 - \frac{1}{2}\sin^2(2x)}\,dx = \int \frac{2\cos 2x}{2 - \sin^2(2x)}\,dx.$$

**Step 2: Substitution.** **Sub:** $\boldsymbol{u = \sin(2x) \Rightarrow du = 2\cos(2x)\,dx}$**.**

$$I_{10} = \int \frac{du}{2 - u^2}.$$

**Step 3: Integrate.** Since $|u| = |\sin 2x| \le 1 < \sqrt{2}$, we have $2 - u^2 > 0$ always. Use $\displaystyle\int \frac{du}{a^2 - u^2} = \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right|$ with $a = \sqrt{2}$:

$$I_{10} = \frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2}+u}{\sqrt{2}-u}\right| = \frac{\sqrt{2}}{4}\ln\left(\frac{\sqrt{2}+\sin 2x}{\sqrt{2}-\sin 2x}\right) + C.$$

(Absolute value dropped since both factors are positive for all $x$.)

$$\boxed{I_{10} = \frac{\sqrt{2}}{4}\ln\left(\frac{\sqrt{2}+\sin 2x}{\sqrt{2}-\sin 2x}\right) + C}$$

**▸ Common Pitfall:**

Do not attempt Weierstrass or Bioche here — the form $\sin^4 x + \cos^4 x$ is algebraic, not directly a rational function of sin and cos. The trig identity step is mandatory before any substitution is visible.

11. $I_{11} = \displaystyle\int \frac{x^2}{(x+1)\sqrt{x-1}}\,dx$ **[Tricky]**

***Method:*** *Radical elimination via direct substitution.*

***Why we do this:***

The radical $\sqrt{x-1}$ is the only irrational term. Setting $u = \sqrt{x-1}$ eliminates it directly and converts the rational factor $(x+1)$ into $u^2 + 2$, giving a polynomial integrand after simplification.

**Execution:** **Sub:** $\boldsymbol{u = \sqrt{x-1} \Rightarrow x = u^2 + 1, \quad dx = 2u\,du.}$

$$I_{11} = \int \frac{(u^2+1)^2}{(u^2+2)\cdot u} \cdot 2u\,du = 2\int \frac{u^4 + 2u^2 + 1}{u^2 + 2}\,du.$$

Long division: $\dfrac{u^4+2u^2+1}{u^2+2} = u^2 + \dfrac{1}{u^2+2}$.

$$I_{11} = 2\int\left(u^2 + \frac{1}{u^2+2}\right)du = 2\left(\frac{u^3}{3} + \frac{1}{\sqrt{2}}\arctan\frac{u}{\sqrt{2}}\right).$$

Back-substitute $u = \sqrt{x-1}$:

$$\boxed{I_{11} = \frac{2}{3}(x-1)^{3/2} + \sqrt{2}\arctan\sqrt{\frac{x-1}{2}} + C}$$

**▶ Common Pitfall:**

Verify: $u^4+2u^2+1 = (u^2+1)^2$ and $u^2+2 \neq u^2+1$. The long division remainder is 1, not 0 — the integrand is *not* a polynomial and the $\frac{1}{u^2+2}$ term must not be dropped.

## Category 3: Rational Fractions

12. $I_{12} = \displaystyle\int_0^1 \frac{x}{x^3+x^2+x+1}\,dx$ **[Standard]**

***Method:*** *Factor by grouping, then PFD.*

***Why we do this:***

The denominator is a degree-3 polynomial. Before partial fractions, always try to factor. Grouping $x^3+x^2+x+1 = x^2(x+1)+(x+1) = (x+1)(x^2+1)$ reveals one linear and one irreducible quadratic factor, giving the canonical PFD form.

**Execution:** $x^3+x^2+x+1 = (x+1)(x^2+1)$. PFD:

$$\frac{x}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}.$$

Multiply through: $x = A(x^2+1)+(Bx+C)(x+1)$. Plug in $x=-1$: $-1 = 2A \Rightarrow A = -\frac{1}{2}$. Compare $x^2$: $0 = A+B \Rightarrow B = \frac{1}{2}$. Compare constants: $0 = A+C \Rightarrow C = \frac{1}{2}$.

$$I_{12} = \Big[-\tfrac{1}{2}\ln(x+1) + \tfrac{1}{4}\ln(x^2+1) + \tfrac{1}{2}\arctan x\Big]_0^1.$$

At $x=1$: $-\frac{\ln 2}{2} + \frac{\ln 2}{4} + \frac{\pi}{8} = -\frac{\ln 2}{4} + \frac{\pi}{8}$. At $x=0$: 0.

$$\boxed{I_{12} = \frac{\pi}{8} - \frac{\ln 2}{4}}$$

**✓ Sanity Check:**

At $x=0$: all three terms vanish. This is a free check — if your antiderivative does not give 0 at $x=0$, the PFD coefficients are wrong.

13. $I_{13} = \displaystyle\int \frac{x^2+1}{x^4+1}\,dx$ **[Tricky]**

***Method:*** *Divide numerator and denominator by $x^2$, then substitute $u = x - 1/x$.*

***Why we do this:***

$x^4+1$ does not factor over $\mathbb{R}$ in an obvious way. Dividing by $x^2$ creates $(x-1/x)^2+2$ in the denominator because $x^2+1/x^2=(x-1/x)^2+2$. The numerator $(1+1/x^2)\,dx$ is then exactly $d(x-1/x)$.

**Execution:**

$$I_{13}=\int\frac{1+1/x^2}{x^2+1/x^2}\,dx=\int\frac{1+1/x^2}{(x-1/x)^2+2}\,dx.$$

**Sub:** $\boldsymbol{u=x-\frac{1}{x}\Rightarrow du=\left(1+\frac{1}{x^2}\right)dx.}$

$$I_{13}=\int\frac{du}{u^2+2}=\frac{1}{\sqrt{2}}\arctan\frac{u}{\sqrt{2}}.$$

Back-substitute $u=x-1/x$:

$$\boxed{I_{13}=\frac{1}{\sqrt{2}}\arctan\left(\frac{x^2-1}{x\sqrt{2}}\right)+C}$$

**► Common Pitfall:**

Note $u=x-1/x$, not $x+1/x$. The sign comes from grouping $x^2+1/x^2=(x-1/x)^2+2$. Using $x+1/x$ gives $(x+1/x)^2-2$, which is the setup for $I_{13}$'s companion integral $\int\frac{x^2-1}{x^4+1}\,dx$ instead.

14. $I_{14}=\displaystyle\int\frac{1}{x^3+1}\,dx$ **[Standard]**

***Method:*** *Standard PFD after factoring via sum of cubes.*

***Why we do this:***

$x^3+1=(x+1)(x^2-x+1)$ by the sum of cubes formula. The quadratic $x^2-x+1$ has discriminant $\Delta=1-4=-3<0$, confirming it is irreducible over $\mathbb{R}$. Complete the square to integrate the quadratic part via arctan.

**Execution:**

$$\frac{1}{(x+1)(x^2-x+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}.$$

At $x=-1$: $1=3A\Rightarrow A=\frac{1}{3}$. Compare $x^2$: $0=A+B\Rightarrow B=-\frac{1}{3}$. Compare constants: $1=A+C\Rightarrow C=\frac{2}{3}$.

Complete the square: $x^2-x+1=\left(x-\frac{1}{2}\right)^2+\frac{3}{4}$.

$$\int\frac{-x/3+2/3}{x^2-x+1}\,dx=-\frac{1}{6}\int\frac{2x-1}{x^2-x+1}\,dx+\frac{1}{2}\int\frac{dx}{(x-\frac{1}{2})^2+\frac{3}{4}}.$$

$$=-\frac{1}{6}\ln(x^2-x+1)+\frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}.$$

$$\boxed{I_{14}=\frac{1}{3}\ln|x+1|-\frac{1}{6}\ln(x^2-x+1)+\frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}+C}$$

**✓ Sanity Check:**

Sanity check at $x = 0$: the integrand gives $\frac{1}{1} = 1$. Differentiate your answer at $x = 0$ and verify it equals 1. The derivative of $\frac{1}{3}\ln|x+1|$ at 0 is $\frac{1}{3}$; of $-\frac{1}{6}\ln(x^2 - x + 1)$ is $+\frac{1}{6}$; of $\frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}$ is $\frac{2}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}\cdot(1+1/3)} = \frac{1}{2}$. Sum: $\frac{1}{3} + \frac{1}{6} + \frac{1}{2} = 1$. ✓

15. $I_{15} = \displaystyle\int \frac{3x+1}{(x-1)^2(x^2+1)}\,dx$ **[Time-Sink]**

***Method:*** *PFD with repeated linear and irreducible quadratic factors.*

***Why we do this:***

A repeated linear factor $(x-1)^2$ requires two terms: $\frac{A}{x-1}$ and $\frac{B}{(x-1)^2}$. The irreducible quadratic $x^2+1$ requires $\frac{Cx+D}{x^2+1}$. The coefficients must be found by systematic matching — shortcuts fail here.

**Execution:**

$$\frac{3x+1}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}.$$

Multiply through by $(x-1)^2(x^2+1)$:

$$3x + 1 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2.$$

At $x = 1$: $4 = 2B \Rightarrow B = 2$. At $x = 0$: $1 = -A + B + D \Rightarrow D = A - 1$. Compare $x^3$: $0 = A + C$. Compare $x^2$: $0 = -A + B - 2C + D = -A + 2 - 2C + A - 1 = 1 - 2C \Rightarrow C = \frac{1}{2}$. So $A = -\frac{1}{2}$, $D = -\frac{3}{2}$.

Integrating each term:

$$\int \frac{-1/2}{x-1}\,dx = -\frac{1}{2}\ln|x-1|, \qquad \int \frac{2}{(x-1)^2}\,dx = -\frac{2}{x-1},$$

$$\int \frac{\frac{1}{2}x - \frac{3}{2}}{x^2+1}\,dx = \frac{1}{4}\ln(x^2+1) - \frac{3}{2}\arctan x.$$

$$\boxed{I_{15} = -\frac{1}{2}\ln|x-1| - \frac{2}{x-1} + \frac{1}{4}\ln(x^2+1) - \frac{3}{2}\arctan x + C}$$

**▶ Common Pitfall:**

**This is a known source of systematic errors.** A common wrong answer is $-\frac{3}{2}\ln|x-1| + \frac{3}{4}\ln(x^2+1) - \frac{1}{2}\arctan x - \frac{2}{x-1}$, obtained by using incorrect coefficients $A = -3/2$, $C = 3/4$. These come from misreading the linear system. Always substitute $x = 1$ first to pin $B$, then use $x = 0$ and coefficient comparison for the rest.

**✓ Sanity Check:**

At $x = 0$: integrand $= \frac{1}{1\cdot 1} = 1$. Differentiate your answer at $x = 0$: contribution from $-\frac{1}{2}\ln|x-1|$ is $\frac{1}{2}$; from $-\frac{2}{x-1}$ is $\frac{2}{(x-1)^2}\Big|_0 = 2$; from $\frac{1}{4}\ln(x^2+1)$ is 0; from $-\frac{3}{2}\arctan x$ is $-\frac{3}{2}$. Sum: $\frac{1}{2} + 2 + 0 - \frac{3}{2} = 1$. ✓

## Category 4: Bioche's Rules

16. $I_{16} = \displaystyle\int \frac{\sin^3 x}{2+\cos x}\,dx$ **[Standard]**

***Method:*** *Bioche test $\omega(-x) = \omega(x)$: substitute $u = \cos x$.*

> ***Why we do this:***
>
> Bioche's rule 1: if replacing $x \to -x$ leaves the differential form $f(\sin x, \cos x)\,dx$ unchanged, use $u = \cos x$. Here $\sin^3(-x) = -\sin^3 x$ and $d(-x) = -dx$, so the product gives the same form. The substitution then converts the whole expression to a rational function in $u$.

**Execution: Sub: $u = \cos x \Rightarrow du = -\sin x\,dx, \quad \sin^2 x = 1 - u^2$.**

$$I_{16} = \int \frac{\sin^2 x \cdot \sin x\,dx}{2+\cos x} = \int \frac{(1-u^2)(-du)}{2+u} = \int \frac{u^2-1}{u+2}\,du.$$

Long division: $\dfrac{u^2-1}{u+2} = u - 2 + \dfrac{3}{u+2}$.

$$I_{16} = \frac{u^2}{2} - 2u + 3\ln|u+2| + C.$$

$$\boxed{I_{16} = \frac{\cos^2 x}{2} - 2\cos x + 3\ln(2+\cos x) + C}$$

> **▶ Common Pitfall:**
>
> The substitution gives $du = -\sin x\,dx$, so $\sin x\,dx = -du$. Students often forget the negative sign, producing a wrong sign in every term. Write $\boldsymbol{du = -\sin x\,dx}$ in bold before substituting.

17. $I_{17} = \displaystyle\int \frac{\cos x}{\sin^2 x - 5\sin x + 6}\,dx$ **[Standard]**

***Method:*** *Bioche test $\omega(\pi - x) = \omega(x)$: substitute $u = \sin x$.*

> ***Why we do this:***
>
> Bioche's rule 2 applies when the form is invariant under $x \to \pi - x$. Here $\sin(\pi - x) = \sin x$, $\cos(\pi - x) = -\cos x$, and $d(\pi - x) = -dx$: the two sign changes cancel, leaving the form unchanged. The substitution $u = \sin x$ then makes the denominator a factorable quadratic in $u$.

**Execution: Sub: $u = \sin x \Rightarrow du = \cos x\,dx$.**

$$I_{17} = \int \frac{du}{u^2-5u+6} = \int \frac{du}{(u-3)(u-2)}.$$

PFD: $\dfrac{1}{(u-3)(u-2)} = \dfrac{1}{u-3} - \dfrac{1}{u-2}$.

$$I_{17} = \ln|u-3| - \ln|u-2| = \ln\left|\frac{u-3}{u-2}\right|.$$

$$\boxed{I_{17} = \ln\left|\frac{\sin x - 3}{\sin x - 2}\right| + C}$$

**► Common Pitfall:**

Since $\sin x \in [-1, 1]$, we have $\sin x - 3 < 0$ and $\sin x - 2 < 0$ always. So $\frac{\sin x - 3}{\sin x - 2} > 0$ and the absolute value is redundant here — but writing it is still correct and safe.

18. $I_{18} = \displaystyle\int \frac{1}{1+\sin^2 x}\,dx$ **[Tricky]**

***Method:*** *Bioche test* $\omega(\pi + x) = \omega(x)$*: substitute* $t = \tan x$.

***Why we do this:***

Bioche's rule 3: if the form is $\pi$-periodic, use $t = \tan x$. This is equivalent to the shortcut: *only even powers of* $\sin$ *and* $\cos$ $\Rightarrow$ jump directly to $t = \tan x$. The substitution converts $\sin^2 x = \frac{t^2}{1+t^2}$ and eliminates the $dx$ via $dx = \frac{dt}{1+t^2}$, giving a rational integrand in $t$.

**Execution:** **Sub:** $\boldsymbol{t = \tan x \Rightarrow dx = \dfrac{dt}{1+t^2}, \quad \sin^2 x = \dfrac{t^2}{1+t^2}}$.

$$I_{18} = \int \frac{1}{1+\frac{t^2}{1+t^2}} \cdot \frac{dt}{1+t^2} = \int \frac{1+t^2}{1+2t^2} \cdot \frac{dt}{1+t^2} = \int \frac{dt}{1+2t^2}.$$

$$= \frac{1}{\sqrt{2}} \arctan(\sqrt{2}\,t) + C.$$

$$\boxed{I_{18} = \frac{1}{\sqrt{2}} \arctan(\sqrt{2}\tan x) + C}$$

**► Common Pitfall:**

The result is $\frac{1}{\sqrt{2}}\arctan(\sqrt{2}\tan x)$, not $\frac{1}{\sqrt{2}}\arctan\left(\frac{\tan x}{\sqrt{2}}\right)$. Using $\int \frac{dt}{1+2t^2}$: factor out $\frac{1}{2}$ to get $\frac{1}{2}\int \frac{dt}{t^2+\frac{1}{2}} = \frac{1}{2} \cdot \frac{1}{1/\sqrt{2}} \arctan\left(\frac{t}{1/\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\arctan(\sqrt{2}\,t)$.

19. $I_{19} = \displaystyle\int \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\,dx$ **[Tricky]**

***Method:*** *Spot that the numerator is the derivative of* $\sin x - \cos x$*; use that substitution.*

***Why we do this:***

Notice $(\sin x - \cos x)' = \cos x + \sin x$, exactly the numerator. Setting $u = \sin x - \cos x$ makes $du = (\cos x + \sin x)\,dx$. Then $u^2 = (\sin x - \cos x)^2 = 1 - \sin 2x$, so $\sin 2x = 1 - u^2$ clears the radical completely.

**Execution:** **Sub:** $\boldsymbol{u = \sin x - \cos x \Rightarrow du = (\cos x + \sin x)\,dx. \quad u^2 = 1 - \sin 2x \Rightarrow \sin 2x = 1}$

$$I_{19} = \int \frac{du}{\sqrt{1-u^2}} = \arcsin u.$$

$$\boxed{I_{19} = \arcsin(\sin x - \cos x) + C}$$

**▶ Common Pitfall:**

The substitution requires $\sin 2x > 0$ (so that the original integrand is defined). Verify the domain: $\sin 2x = 1 - u^2$ with $|u| \le \sqrt{2}$. When $\sin 2x = 0$, the original integral has a singularity — the result is valid on any open interval where $\sin 2x > 0$.

## Category 5: Roots & Irrationalities

20. $I_{20} = \displaystyle\int \frac{x^3}{\sqrt{x^2+1}}\,dx$ **[Standard]**

***Method:*** *Direct substitution to collapse the radical.*

***Why we do this:***

The radical $\sqrt{x^2+1}$ suggests $u = x^2+1$. Then $x^2 = u-1$ handles the numerator, and $du = 2x\,dx$ absorbs one power of $x$ from $x^3 = x^2 \cdot x$. This reduces the integrand to a power of $u$.

**Execution:** **Sub:** $\boldsymbol{u = x^2 + 1 \Rightarrow du = 2x\,dx, \quad x^2 = u - 1.}$

$$I_{20} = \int \frac{x^2 \cdot x\,dx}{\sqrt{u}} = \frac{1}{2}\int \frac{u-1}{\sqrt{u}}\,du = \frac{1}{2}\int \left(u^{1/2} - u^{-1/2}\right)du = \frac{1}{2}\left(\frac{2u^{3/2}}{3} - 2u^{1/2}\right).$$

$$\boxed{I_{20} = \frac{1}{3}(x^2-2)\sqrt{x^2+1} + C}$$

**▶ Common Pitfall:**

Factor the back-substitution cleanly: $\frac{u^{3/2}}{3} - u^{1/2} = u^{1/2}\left(\frac{u}{3} - 1\right) = \sqrt{x^2+1} \cdot \frac{x^2+1-3}{3} = \frac{(x^2-2)\sqrt{x^2+1}}{3}$. Verify by differentiation before proceeding.

21. $I_{21} = \displaystyle\int \frac{dx}{x\sqrt{6-x-x^2}}$ **[Time-Sink]**

***Method:*** *Reciprocal substitution $x = 1/t$ to bring $x$ inside the radical, then complete the square.*

***Why we do this:***

The difficulty is $x$ appearing both outside and inside the radical. The substitution $x = 1/t$ transfers $x$ into the radical expression, converting the integrand into $-1/\sqrt{6t^2-t-1}$, which is a standard arccosh form after completing the square.

**Execution:** **Sub:** $\boldsymbol{x = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}\,dt.}$ Assume $\boldsymbol{x > 0 \Rightarrow t > 0.}$

$$6 - x - x^2 = 6 - \frac{1}{t} - \frac{1}{t^2} = \frac{6t^2 - t - 1}{t^2}, \qquad \sqrt{6 - x - x^2} = \frac{\sqrt{6t^2-t-1}}{t}.$$

$$I_{21} = \int \frac{-dt/t^2}{(1/t)\cdot(\sqrt{6t^2-t-1}/t)} = -\int \frac{dt}{\sqrt{6t^2-t-1}}.$$

Complete the square: $6t^2 - t - 1 = 6\left[\left(t - \frac{1}{12}\right)^2 - \frac{25}{144}\right]$.

$$I_{21} = -\frac{1}{\sqrt{6}} \int \frac{dt}{\sqrt{(t - \frac{1}{12})^2 - (\frac{5}{12})^2}} = -\frac{1}{\sqrt{6}} \operatorname{arccosh}\left(\frac{t - \frac{1}{12}}{\frac{5}{12}}\right) + C.$$

Simplify and back-substitute $t = 1/x$:

$$\boxed{I_{21} = -\frac{1}{\sqrt{6}} \operatorname{arccosh}\left(\frac{12/x - 1}{5}\right) + C}$$

**► Common Pitfall:**

After the substitution, $\sqrt{t^2} = |t| = t$ only because we assumed $t > 0$ (i.e. $x > 0$). If $x < 0$, the sign flips. State the domain assumption explicitly.

22. $I_{22} = \displaystyle\int \frac{\sqrt{x^2 - 1}}{x}\, dx$ **[Tricky]**

***Method:*** *Algebraic substitution* $u = \sqrt{x^2 - 1}$ *to clear the radical.*

***Why we do this:***

Setting $u = \sqrt{x^2 - 1}$ directly names the radical as the new variable. From $u^2 = x^2 - 1$ we get $x^2 = u^2 + 1$, and differentiating gives $u\,du = x\,dx$, i.e. $\frac{u\,du}{u^2+1} = \frac{dx}{x}$. This converts the integrand to a rational function in $u$.

**Execution: Sub:** $\boldsymbol{u = \sqrt{x^2 - 1} \Rightarrow u^2 + 1 = x^2 \Rightarrow \dfrac{u\,du}{u^2 + 1} = \dfrac{dx}{x}}$**.**

$$I_{22} = \int \frac{u}{-} \cdot \frac{u\,du}{u^2 + 1} = \int \frac{u^2}{u^2 + 1}\, du = \int \left(1 - \frac{1}{u^2 + 1}\right) du = u - \arctan u.$$

$$\boxed{I_{22} = \sqrt{x^2 - 1} - \arctan\left(\sqrt{x^2 - 1}\right) + C}$$

**► Common Pitfall:**

Do not confuse $\frac{dx}{x}$ and $dx$. Here $\frac{u\,du}{u^2+1} = \frac{dx}{x}$, meaning the entire fraction $\frac{\sqrt{x^2-1}}{x}\, dx = u \cdot \frac{u\,du}{u^2+1} \cdot \frac{x}{x}$... simplify carefully. The substitution is for $dx/x$, not for $dx$ alone.

## Category 6: Universal Substitution (Weierstrass)

23. $I_{23} = \displaystyle\int \frac{1}{2 + \cos x}\, dx$ **[Standard]**

***Method:*** *Weierstrass* $t = \tan(x/2)$ *for pure trigonometric rational forms.*

***Why we do this:***

Bioche's three rules all fail for $\frac{1}{2+\cos x}$ (the form is not invariant under $x \to -x$, $x \to \pi - x$, or $x \to \pi + x$ simultaneously). The universal fallback is Weierstrass: $t = \tan(x/2)$ converts $\sin x$, $\cos x$, and $dx$ into rational functions of $t$, reducing the integral to a standard rational form.

**Execution:** **Sub:** $t = \tan(x/2) \Rightarrow dx = \dfrac{2\,dt}{1+t^2}, \quad \cos x = \dfrac{1-t^2}{1+t^2}.$

$$I_{23} = \int \frac{1}{2+\frac{1-t^2}{1+t^2}} \cdot \frac{2\,dt}{1+t^2} = \int \frac{1+t^2}{2(1+t^2)+(1-t^2)} \cdot \frac{2\,dt}{1+t^2} = \int \frac{2\,dt}{t^2+3}.$$

$$= \frac{2}{\sqrt{3}} \arctan \frac{t}{\sqrt{3}} + C.$$

$$\boxed{I_{23} = \frac{2}{\sqrt{3}} \arctan\left(\frac{\tan(x/2)}{\sqrt{3}}\right) + C}$$

**▶ Common Pitfall:**

The denominator after substitution: $2+\frac{1-t^2}{1+t^2} = \frac{2(1+t^2)+1-t^2}{1+t^2} = \frac{t^2+3}{1+t^2}$, *not* $\frac{3+t^2}{1-t^2}$. A sign error in expanding gives the wrong denominator entirely.

24. $I_{24} = \displaystyle\int \frac{1}{\sin x + \cos x + 1}\, dx$ **[Tricky]**

***Method:*** *Weierstrass substitution; the denominator collapses to a linear form.*

***Why we do this:***

The presence of the $+1$ alongside $\sin x$ and $\cos x$ is the hint that Weierstrass will simplify dramatically. After substitution, $\sin x + \cos x + 1 = \frac{2t+1-t^2+1+t^2}{1+t^2} = \frac{2(t+1)}{1+t^2}$. This cancels with the $(1+t^2)$ in $dx$, leaving just $\int \frac{dt}{t+1}$.

**Execution:** **Sub:** $t = \tan(x/2), \quad \sin x = \dfrac{2t}{1+t^2}, \quad \cos x = \dfrac{1-t^2}{1+t^2}, \quad dx = \dfrac{2\,dt}{1+t^2}.$

Denominator: $\dfrac{2t}{1+t^2} + \dfrac{1-t^2}{1+t^2} + 1 = \dfrac{2t+1-t^2+1+t^2}{1+t^2} = \dfrac{2(t+1)}{1+t^2}.$

$$I_{24} = \int \frac{1}{\frac{2(t+1)}{1+t^2}} \cdot \frac{2\,dt}{1+t^2} = \int \frac{2(1+t^2)}{2(t+1)(1+t^2)}\, dt = \int \frac{dt}{t+1} = \ln|t+1|.$$

$$\boxed{I_{24} = \ln\left|1+\tan\frac{x}{2}\right| + C}$$

**▶ Common Pitfall:**

The cancellation is only valid when $t+1 \neq 0$, i.e. $\tan(x/2) \neq -1$, i.e. $x \neq -\pi/2+2k\pi$. On these excluded points the original integrand has a singularity anyway — state the domain of validity.

## Category 7: Advanced & Mixed Techniques

25. $I_{25} = \displaystyle\int (\sin x \cos x)\, e^{\sin x}\, dx$ **[Standard]**

***Method:*** *Substitution $u = \sin x$, then standard IBP.*

**Why we do this:**

The $\cos x\,dx$ factor is $d(\sin x)$, so setting $u = \sin x$ absorbs it cleanly into $du$. The remaining integrand is $ue^u$, a textbook IBP case.

**Execution: Sub: $\boldsymbol{u = \sin x \Rightarrow du = \cos x\,dx}$.**

$I_{25} = \int ue^u\,du$. IBP: $p = u$, $\boldsymbol{dq = e^u\,du} \Rightarrow dp = du$, $\boldsymbol{q = e^u}$:

$$\int ue^u\,du = ue^u - \int e^u\,du = (u-1)e^u + C.$$

$$\boxed{I_{25} = (\sin x - 1)e^{\sin x} + C}$$

26. $I_{26} = \displaystyle\int_0^{\pi/4} \ln(1 + \tan x)\,dx$ **[Tricky]**

***Method:*** *King's Property: $x \mapsto \frac{\pi}{4} - x$.*

**Why we do this:**

The King's Property converts a definite integral over $[a, b]$ to one with complementary argument. Here it changes $\tan x$ to $\tan(\pi/4 - x) = \frac{1-\tan x}{1+\tan x}$, so $1 + \tan(\pi/4 - x) = \frac{2}{1+\tan x}$. The logarithm splits as $\ln 2 - \ln(1 + \tan x)$, and adding the two forms of $I_{26}$ isolates it.

**Execution:** Apply King's property $x \mapsto \frac{\pi}{4} - x$:

$$I_{26} = \int_0^{\pi/4} \ln\left(1 + \frac{1-\tan x}{1+\tan x}\right) dx = \int_0^{\pi/4} \ln\left(\frac{2}{1+\tan x}\right) dx = \int_0^{\pi/4} \ln 2\,dx - I_{26}.$$

$$2I_{26} = \frac{\pi}{4}\ln 2 \implies \boxed{I_{26} = \frac{\pi}{8}\ln 2}$$

27. $I_{27} = \displaystyle\int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx$ **[Tricky]**

***Method:*** *Trigonometric substitution $x = \tan\theta$ reveals $I_{26}$.*

**Why we do this:**

The denominator $x^2 + 1$ screams $x = \tan\theta$: it cancels with the $(1 + \tan^2\theta)$ from $dx$. The bounds $[0, 1]$ map to $[0, \pi/4]$, and the numerator $\ln(\tan\theta + 1)$ is exactly the integrand of $I_{26}$.

**Execution: Sub: $\boldsymbol{x = \tan\theta \Rightarrow dx = (1 + \tan^2\theta)\,d\theta}$.** Bounds: $\boldsymbol{0 \to \pi/4}$.

$$I_{27} = \int_0^{\pi/4} \frac{\ln(\tan\theta + 1)}{1 + \tan^2\theta} \cdot (1 + \tan^2\theta)\,d\theta = \int_0^{\pi/4} \ln(1 + \tan\theta)\,d\theta = I_{26}.$$

$$\boxed{I_{27} = \frac{\pi}{8}\ln 2}$$

28. $I_{28} = \displaystyle\int_0^{\pi/2} \ln(2\cos x)\,dx$ **[Tricky]**

***Method:*** *King's Property then logarithmic addition to create a self-referential equation.*

***Why we do this:***

Applying King's property swaps $\cos x$ and $\sin x$. Adding both forms gives $\ln(4\sin x\cos x) = \ln(2\sin 2x)$. A further substitution $u = 2x$ re-expresses the integral over $[0, \pi]$, which by symmetry equals $2I_{28}$, forcing $I_{28} = 0$.

**Execution:** $x \mapsto \frac{\pi}{2} - x$: $I_{28} = \int_0^{\pi/2} \ln(2\sin x)\,dx$. Adding: $2I_{28} = \int_0^{\pi/2} \ln(4\sin x\cos x)\,dx = \int_0^{\pi/2} \ln(2\sin 2x)\,dx$. **Sub:** $\boldsymbol{u = 2x \Rightarrow du = 2\,dx}$. Bounds: $\boldsymbol{0 \to \pi}$.

$$2I_{28} = \frac{1}{2}\int_0^{\pi} \ln(2\sin u)\,du.$$

By symmetry ($\sin(\pi - u) = \sin u$): $\int_0^{\pi} \ln(2\sin u)\,du = 2\int_0^{\pi/2} \ln(2\sin u)\,du = 2I_{28}$. So $2I_{28} = I_{28}$, giving:

$$\boxed{I_{28} = 0}$$

**► Common Pitfall:**

$I_{28} = 0$ does *not* mean the integrand is zero. $\ln(2\cos x)$ takes negative values for $x$ near $\pi/2$ (where $\cos x \to 0$), and positive values near $x = 0$ (where $2\cos 0 = 2 > 1$). The areas cancel exactly.

29. $I_{29} = \displaystyle\int \frac{3}{x^4 + x}\,dx$ **[Standard]**

***Method:*** *Factor, then use the logarithmic derivative.*

***Why we do this:***

$x^4 + x = x(x^3 + 1)$. Writing $\frac{3}{x(x^3+1)} = \frac{3}{x} - \frac{3x^2}{x^3+1}$ exploits the fact that $\frac{d}{dx}[x^3 + 1] = 3x^2$, making the second term a direct logarithmic integral.

**Execution:**

$$\frac{3}{x(x^3 + 1)} = \frac{3}{x} - \frac{3x^2}{x^3 + 1}.$$

$$\boxed{I_{29} = 3\ln|x| - \ln|x^3 + 1| + C = \ln\left|\frac{x^3}{x^3 + 1}\right| + C}$$

30. $I_{30} = \displaystyle\int \frac{1}{(1 + x)\sqrt{1 + x + x^2}}\,dx$ **[Time-Sink]**

***Method:*** *Euler reciprocal substitution $u = \frac{1}{x+1}$ for (linear)$\cdot\sqrt{\text{quadratic}}$.*

***Why we do this:***

When the integrand has the form $\frac{1}{L(x)\sqrt{Q(x)}}$ where $L$ is linear, setting $u = 1/L(x)$ eliminates $L$ from the denominator and turns $Q$ into a quadratic in $u$ under the radical, which is then handled by completing the square.

**Execution:** **Sub:** $\boldsymbol{u = \dfrac{1}{x + 1} \Rightarrow x = \dfrac{1 - u}{u}, \quad dx = -\dfrac{du}{u^2}.}$

$$1+x+x^2=\frac{3u^2-3u+1}{u^2},\qquad \sqrt{1+x+x^2}=\frac{\sqrt{3u^2-3u+1}}{u}.$$

$$I_{30}=\int \frac{1}{\frac{1}{u}\cdot\frac{\sqrt{3u^2-3u+1}}{u}}\cdot\left(-\frac{du}{u^2}\right)=-\int\frac{du}{\sqrt{3u^2-3u+1}}.$$

Complete the square: $3u^2-3u+1=3\left(u-\frac{1}{2}\right)^2+\frac{1}{4}$. This is arcsinh form. Back-substitute $u=\frac{1}{x+1}$, giving $1-2u=\frac{x-1}{x+1}$:

$$\boxed{I_{30}=-\operatorname{arcsinh}\left(\frac{1-x}{(x+1)\sqrt{3}}\right)+C}$$

31. $I_{31}=\displaystyle\int_0^{\pi/2}\frac{\sin^3 x}{\sin^3 x+\cos^3 x}\,dx$ **[Standard]**

***Method:*** *King's Property:* $x\mapsto \pi/2-x$.

***Why we do this:***

King's property swaps $\sin x\leftrightarrow\cos x$. The denominator $\sin^3 x+\cos^3 x$ is symmetric under this swap, so the complementary integral has $\cos^3 x$ in the numerator. Adding the two forms telescopes the denominator.

**Execution:** $x\mapsto\frac{\pi}{2}-x$: $I_{31}=\displaystyle\int_0^{\pi/2}\frac{\cos^3 x}{\cos^3 x+\sin^3 x}\,dx$. Adding: $2I_{31}=\displaystyle\int_0^{\pi/2}1\,dx=\frac{\pi}{2}$.

$$\boxed{I_{31}=\frac{\pi}{4}}$$

32. $I_{32}=\displaystyle\int\frac{1}{x}\sqrt{\frac{1-x}{1+x}}\,dx$ **[Tricky]**

***Method:*** *Rationalise by multiplying by* $\sqrt{1-x}/\sqrt{1-x}$, *then split.*

***Why we do this:***

Multiplying top and bottom by $\sqrt{1-x}$ converts the radical: $\sqrt{\frac{1-x}{1+x}}\cdot\frac{\sqrt{1-x}}{\sqrt{1-x}}=\frac{1-x}{\sqrt{1-x^2}}$. The resulting two integrals are both standard: one is arccosh type and the other is arcsin.

**Execution:**

$$I_{32}=\int\frac{1-x}{x\sqrt{1-x^2}}\,dx=\int\frac{dx}{x\sqrt{1-x^2}}-\int\frac{dx}{\sqrt{1-x^2}}.$$

First integral: $\displaystyle\int\frac{dx}{x\sqrt{1-x^2}}=-\operatorname{arccosh}\frac{1}{|x|}$. Second: $\displaystyle\int\frac{dx}{\sqrt{1-x^2}}=\arcsin x$.

$$\boxed{I_{32}=-\operatorname{arccosh}\left(\frac{1}{|x|}\right)-\arcsin x+C}$$

**▶ Common Pitfall:**

The domain requires $0<x\le 1$ (so that $\frac{1-x}{1+x}\ge 0$ and $\frac{1}{|x|}\ge 1$ for arccosh). State the domain explicitly.

# Category 8: Theoretical Problems (Detailed Proofs)

**Theoretical Applications & Synthesis.** This section provides rigorous, step-by-step proofs for the generalised integral theorems and recursive sequences. Precision of phrasing is graded — define every function and state every theorem you invoke.

33. **Problem I (The King's Property):**

    (a) Let $u = a + b - x \Rightarrow du = -dx$. When $x = a \Rightarrow u = b$; $x = b \Rightarrow u = a$:

    $$\int_a^b f(x)\,dx = \int_b^a f(a+b-u)(-du) = \int_a^b f(a+b-u)\,du.$$

    Since $u$ is a dummy variable, we may rename it $x$. ☐

    (b) Let $I = \displaystyle\int_0^{\pi/2} \frac{g(\sin x)}{g(\sin x) + g(\cos x)}\,dx$. Apply part (a) with $a = 0$, $b = \pi/2$, so $a+b-x = \pi/2 - x$. Under $x \mapsto \frac{\pi}{2} - x$: $\sin x \mapsto \sin(\pi/2 - x) = \cos x$ and $\cos x \mapsto \sin x$:

    $$I = \int_0^{\pi/2} \frac{g(\cos x)}{g(\cos x) + g(\sin x)}\,dx.$$

    Adding the two expressions for $I$:

    $$2I = \int_0^{\pi/2} \frac{g(\sin x) + g(\cos x)}{g(\sin x) + g(\cos x)}\,dx = \int_0^{\pi/2} 1\,dx = \frac{\pi}{2} \implies \boxed{I = \frac{\pi}{4}}. \quad \square$$

    (c) We compute $J = \displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}\left(1 + e^{1-2x^2}\right)}\,dx$.

    **Step 1: Substitution $x = \sin\theta$.** **Sub: $\boldsymbol{x = \sin\theta, \quad dx = \cos\theta\,d\theta, \quad \sqrt{1-x^2} = \cos\theta}$.** Bound

    $$J = \int_0^{\pi/2} \frac{\cos\theta\,d\theta}{\cos\theta \cdot \left(1 + e^{1-2\sin^2\theta}\right)} = \int_0^{\pi/2} \frac{d\theta}{1 + e^{\cos 2\theta}},$$

    using $1 - 2\sin^2\theta = \cos(2\theta)$.

    **Step 2: King's property.** Apply $\theta \mapsto \frac{\pi}{2} - \theta$:

    $$J = \int_0^{\pi/2} \frac{d\theta}{1 + e^{\cos(\pi - 2\theta)}} = \int_0^{\pi/2} \frac{d\theta}{1 + e^{-\cos 2\theta}}.$$

    **Step 3: Sum.**

    $$2J = \int_0^{\pi/2} \left(\frac{1}{1 + e^{\cos 2\theta}} + \frac{1}{1 + e^{-\cos 2\theta}}\right) d\theta = \int_0^{\pi/2} 1\,d\theta = \frac{\pi}{2}.$$

    (The two fractions sum to 1 since $\frac{1}{1+e^s} + \frac{1}{1+e^{-s}} = 1$ for all $s \in \mathbb{R}$.)

    **Alternative via (b):** Identify $g(t) = e^{t^2}$. Then $\frac{g(\sin\theta)}{g(\sin\theta)+g(\cos\theta)} = \frac{e^{\sin^2\theta}}{e^{\sin^2\theta}+e^{\cos^2\theta}} = \frac{1}{1+e^{\cos 2\theta}}$, so by (b), $J = \pi/4$.

    $$\boxed{J = \frac{\pi}{4}}$$

34. **Problem II (Logarithmic Sequences):**

    (a) Let $W_n = \displaystyle\int_1^e (\ln x)^n \, dx$. IBP: $u = (\ln x)^n$, $\boldsymbol{dv = dx}$, so $du = \frac{n(\ln x)^{n-1}}{x}\, dx$, $\boldsymbol{v = x}$:

    $$W_n = \big[x(\ln x)^n\big]_1^e - n\int_1^e (\ln x)^{n-1}\, dx = \big(e\cdot 1 - 1\cdot 0\big) - nW_{n-1}.$$

    $$\boxed{W_n = e - nW_{n-1}} \quad \square$$

    (b) For $x \in [1, e]$, $0 \le \ln x \le 1 \Rightarrow (\ln x)^n \ge 0 \Rightarrow W_n \ge 0$. Moreover $\ln x \le 1 \Rightarrow (\ln x)^{n+1} \le (\ln x)^n$; integrating gives $W_{n+1} \le W_n$. The sequence is **strictly decreasing** and bounded below by 0, hence convergent.

    (c) $(W_n)$ converges to $L \ge 0$. From the recurrence: $nW_{n-1} = e - W_n \xrightarrow{n\to\infty} e - L$. If $L > 0$, then $nW_{n-1} \ge n \cdot L \to +\infty$, contradicting the finiteness of $e - L$. Therefore $L = 0$:

    $$\boxed{\lim_{n\to+\infty} W_n = 0.} \quad \square$$

35. **Problem III (The Cyclic IBP System):**

    (a) Evaluate $J = \displaystyle\int e^{\alpha x}\cos(\beta x)\, dx$. IBP with $u = \cos(\beta x)$, $\boldsymbol{dv = e^{\alpha x} dx}$:

    $$J = \frac{1}{\alpha}e^{\alpha x}\cos(\beta x) + \frac{\beta}{\alpha}K, \quad \text{where } K = \int e^{\alpha x}\sin(\beta x)\, dx.$$

    IBP on $K$ with $u = \sin(\beta x)$, $\boldsymbol{dv = e^{\alpha x} dx}$:

    $$K = \frac{1}{\alpha}e^{\alpha x}\sin(\beta x) - \frac{\beta}{\alpha}J.$$

    The linear system relating $J$ and $K$ is:

    $$\begin{cases} \alpha J - \beta K = e^{\alpha x}\cos(\beta x) \\ \beta J + \alpha K = e^{\alpha x}\sin(\beta x) \end{cases}$$

    (b) Multiply the first equation by $\alpha$, the second by $\beta$, and add: $(\alpha^2+\beta^2)J = e^{\alpha x}(\alpha\cos\beta x + \beta\sin\beta x)$. Multiply the first by $-\beta$, the second by $\alpha$, and add: $(\alpha^2+\beta^2)K = e^{\alpha x}(\alpha\sin\beta x - \beta\cos\beta x)$.

    $$\boxed{J = \frac{e^{\alpha x}}{\alpha^2+\beta^2}\big(\alpha\cos(\beta x) + \beta\sin(\beta x)\big) + C, \quad K = \frac{e^{\alpha x}}{\alpha^2+\beta^2}\big(\alpha\sin(\beta x) - \beta\cos(\beta x)\big) + C.}$$

    (c) For $I_2 = \displaystyle\int \cos(\ln x)\, dx$, set $x = e^t \Rightarrow \boldsymbol{dx = e^t\, dt}$. The integral becomes $\displaystyle\int e^t\cos t\, dt = J$ with $\alpha = \beta = 1$:

    $$\int e^t \cos t\, dt = \frac{e^t}{2}(\cos t + \sin t) + C.$$

    Back-substitute $t = \ln x$ (so $e^t = x$, $\cos t = \cos(\ln x)$, $\sin t = \sin(\ln x)$):

    $$\boxed{I_2 = \frac{x}{2}\big(\cos(\ln x) + \sin(\ln x)\big) + C.}$$

    This confirms the result from Category 1, problem 2. $\square$

36. **Problem IV (The Universal Root Trick):**

   (a) **Sub: $t = \sqrt{1+x^n} \Rightarrow t^2 = 1 + x^n \Rightarrow 2t\,dt = nx^{n-1}\,dx$.**
   Rewrite $A_n$ by multiplying the integrand by $\frac{x^{n-1}}{x^{n-1}}$:

$$A_n = \int \frac{x^{n-1}}{x^n\sqrt{1+x^n}}\,dx.$$

   Substitute $x^{n-1}\,dx = \frac{2t\,dt}{n}$, $x^n = t^2-1$, $\sqrt{1+x^n} = t$:

$$A_n = \int \frac{\frac{2t}{n}\,dt}{(t^2-1)\cdot t} = \boxed{\frac{2}{n}\int \frac{dt}{t^2-1}.} \quad \square$$

   (b) $\displaystyle\int \frac{dt}{t^2-1} = \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|$. Therefore:

$$\boxed{A_n(x) = \frac{1}{n}\ln\left|\frac{\sqrt{1+x^n}-1}{\sqrt{1+x^n}+1}\right| + C.}$$

   (c) Let $\Omega_n = \displaystyle\int_1^2 \frac{dx}{x\sqrt{1+x^n}}$. Evaluate using part (b):

$$\Omega_n = \frac{1}{n}\left[\ln\left(\frac{\sqrt{1+2^n}-1}{\sqrt{1+2^n}+1}\right) - \ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right].$$

   As $n \to +\infty$: $\sqrt{1+2^n} \to +\infty$, so $\frac{\sqrt{1+2^n}-1}{\sqrt{1+2^n}+1} \to 1$ and the first log $\to 0$. The factor $\frac{1}{n} \to 0$ forces the second term to 0 as well. Hence $\boxed{\lim_{n\to+\infty} \Omega_n = 0}$. $\square$

37. **Problem V (Asymptotic Tangent Sequence):**

   (a) $J_n + J_{n+2} = \displaystyle\int_0^{\pi/4} \tan^n x(1+\tan^2 x)\,dx$. Since $\frac{d}{dx}[\tan x] = 1 + \tan^2 x = \sec^2 x$:

$$J_n + J_{n+2} = \left[\frac{\tan^{n+1} x}{n+1}\right]_0^{\pi/4} = \boxed{\frac{1}{n+1}.} \quad \square$$

   (b) For $x \in [0, \pi/4]$: $0 \le \tan x \le 1$, so $\tan^n x \ge 0 \Rightarrow J_n \ge 0$. Since $\tan^{n+1} x \le \tan^n x$, integrating gives $J_{n+1} \le J_n$: the sequence is **strictly decreasing** and bounded below by 0, hence convergent.

   (c) Since $J_{n+2} \le J_n$: $2J_{n+2} \le J_n + J_{n+2} = \frac{1}{n+1}$, so $0 \le J_{n+2} \le \frac{1}{2(n+1)} \to 0$. By the Squeeze Theorem: $\boxed{\lim_{n\to+\infty} J_n = 0}$.

   (d) From $J_n + J_{n+2} = \frac{1}{n+1}$ and $J_{n+2} \le J_n$: $2J_n \ge \frac{1}{n+1} \Rightarrow J_n \ge \frac{1}{2(n+1)}$. Shifting: $J_{n-2} + J_n = \frac{1}{n-1}$ and $J_n \le J_{n-2}$: $2J_n \le \frac{1}{n-1} \Rightarrow J_n \le \frac{1}{2(n-1)}$.

$$\boxed{\frac{1}{2(n+1)} \le J_n \le \frac{1}{2(n-1)}, \quad n \ge 2.} \quad \square$$

   (e) Multiply by $2n$: $\frac{2n}{2n+2} \le 2nJ_n \le \frac{2n}{2n-2}$. Both bounds $\to 1$ as $n \to \infty$:

$$\boxed{J_n \sim \frac{1}{2n}.}$$

38. **Problem VI (The Wallis Integrals):**

    (a) Write $I_n = \displaystyle\int_0^{\pi/2} \sin^{n-1} x \cdot \sin x \, dx$. IBP: $u = \sin^{n-1} x$, $\boldsymbol{dv = \sin x \, dx}$, so $du = (n-1)\sin^{n-2} x \cos x \, dx$, $\boldsymbol{v = -\cos x}$:

    $$I_n = \left[-\sin^{n-1} x \cos x\right]_0^{\pi/2} + (n-1)\int_0^{\pi/2} \sin^{n-2} x \cos^2 x \, dx.$$

    The boundary term vanishes at both ends. Using $\cos^2 x = 1 - \sin^2 x$:

    $$I_n = (n-1)(I_{n-2} - I_n) \implies nI_n = (n-1)I_{n-2}.$$

    $$\boxed{I_n = \frac{n-1}{n} I_{n-2}.} \quad \square$$

    (b) $\sin x > 0$ on $(0, \pi/2) \Rightarrow I_n > 0$. Since $\sin x < 1$: $\sin^{n+1} x < \sin^n x$, so $I_{n+1} < I_n$. The sequence is **strictly decreasing**.

    (c) Let $L_n = nI_nI_{n-1}$. Using $nI_n = (n-1)I_{n-2}$:

    $$L_n = (n-1)I_{n-2}I_{n-1} = L_{n-1}.$$

    $(L_n)$ is constant. $L_1 = 1 \cdot I_1 \cdot I_0 = 1 \cdot 1 \cdot \frac{\pi}{2} = \frac{\pi}{2}$.

    $$\boxed{nI_nI_{n-1} = \frac{\pi}{2} \text{ for all } n \geq 1.}$$

    (d) $(I_n)$ decreasing $\Rightarrow I_n \leq I_{n-1} \leq I_{n-2}$. Dividing by $I_n > 0$: $1 \leq \frac{I_{n-1}}{I_n} \leq \frac{I_{n-2}}{I_n} = \frac{n}{n-1}$. By Squeeze: $\frac{I_{n-1}}{I_n} \to 1$. From $nI_n \cdot I_{n-1} = \frac{\pi}{2}$: $nI_n^2 \cdot \frac{I_{n-1}}{I_n} = \frac{\pi}{2}$, so $nI_n^2 \to \frac{\pi}{2}$:

    $$\boxed{I_n \sim \sqrt{\frac{\pi}{2n}}.}$$

    (This is Stirling's approximation for the Wallis sequence, connected to $n! \sim \sqrt{2\pi n}(n/e)^n$.)

39. **Problem VII (Synthesis of Theorems):**

    (a) Define $h(x) = f(x)e^{\lambda x}$. Since $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$, so is $h$. **Boundary conditions:** $h(a) = f(a)e^{\lambda a} = 0 \cdot e^{\lambda a} = 0$ and $h(b) = f(b)e^{\lambda b} = 0 \cdot e^{\lambda b} = 0$. By **Rolle's Theorem**, $\exists\, c \in (a, b)$ such that $h'(c) = 0$. Computing: $h'(x) = e^{\lambda x}\big(f'(x) + \lambda f(x)\big)$. Since $e^{\lambda c} \neq 0$:

    $$\boxed{f'(c) + \lambda f(c) = 0.} \quad \square$$

    (b) Define $G(x) = \displaystyle\int_0^x \big(g(t) - t\big)\, dt$. Then:

    - $G(0) = 0$.
    - $G(1) = \displaystyle\int_0^1 g(t)\, dt - \int_0^1 t \, dt = \frac{1}{2} - \frac{1}{2} = 0$.

    $G$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$ with $G(0) = G(1) = 0$. By **Rolle's Theorem**, $\exists\, \alpha \in (0, 1)$ such that $G'(\alpha) = 0$. By the Fundamental Theorem of Calculus: $G'(x) = g(x) - x$, so $g(\alpha) - \alpha = 0$:

    $$\boxed{g(\alpha) = \alpha.} \quad \square$$

(c) *Second Mean Value Theorem:*

i. Define $V(x) = \int_a^x v(t)\,dt$, so $V(a) = 0$ and $V'(x) = v(x)$. IBP with $u(x)$ and $dV(x) = v(x)\,dx$:

$$\int_a^b u(x)v(x)\,dx = \big[u(x)V(x)\big]_a^b - \int_a^b u'(x)V(x)\,dx = u(b)V(b) - \int_a^b u'(x)V(x)\,dx. \quad \square$$

ii. Since $u$ is strictly decreasing, $u'(x) \le 0$, so $-u'(x) \ge 0$. Multiplying $m \le V(x) \le M$ by $-u'(x) \ge 0$ and integrating:

$$m\int_a^b (-u'(x))\,dx \le -\int_a^b u'(x)V(x)\,dx \le M\int_a^b (-u'(x))\,dx.$$

Since $\int_a^b (-u')\,dx = u(a) - u(b)$:

$$m\big(u(a) - u(b)\big) \le -\int_a^b u'(x)V(x)\,dx \le M\big(u(a) - u(b)\big). \quad \square$$

iii. Add $u(b)V(b)$ to all parts. Since $m \le V(b) \le M$ and $u(b) > 0$: $m\,u(b) \le u(b)V(b) \le M\,u(b)$. Combining:

$$m\,u(a) \le u(b)V(b) + m(u(a) - u(b)) \le \int_a^b u(x)v(x)\,dx \le u(b)V(b) + M(u(a) - u(b)) \le M\,u(a).$$

$$\boxed{m\,u(a) \le \int_a^b u(x)v(x)\,dx \le M\,u(a).} \quad \square$$

iv. Divide by $u(a) > 0$: $m \le \frac{1}{u(a)}\int_a^b u(x)v(x)\,dx \le M$. $V$ is continuous on $[a,b]$ with $\min V = m$ and $\max V = M$. By the **Intermediate Value Theorem**, $\exists\, c \in [a,b]$ such that $V(c) = \frac{1}{u(a)}\int_a^b u(x)v(x)\,dx$:

$$\boxed{\int_a^b u(x)v(x)\,dx = u(a)\int_a^c v(x)\,dx.} \quad \square$$

40. **Problem VIII: "The Final Boss"** **[Time-Sink]**

(a) ***Method:*** *Weierstrass substitution.*

**Sub:** $\boldsymbol{t = \tan(x/2) \Rightarrow dx = \frac{2\,dt}{1+t^2}, \quad \sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}}$. Bounds: $\boldsymbol{0 \to 1}$.

Compute $2\sin x + \cos x + 3$:

$$\frac{4t}{1+t^2} + \frac{1-t^2}{1+t^2} + 3 = \frac{4t + 1 - t^2 + 3(1+t^2)}{1+t^2} = \frac{2t^2 + 4t + 4}{1+t^2} = \frac{2(t^2 + 2t + 2)}{1+t^2}.$$

Hence $(2\sin x + \cos x + 3)^2 = \frac{4(t^2+2t+2)^2}{(1+t^2)^2}$:

$$\Omega = \int_0^1 \frac{(1+t^2)^2}{4(t^2+2t+2)^2} \cdot \frac{2\,dt}{1+t^2} = \boxed{\frac{1}{2}\int_0^1 \frac{t^2+1}{(t^2+2t+2)^2}\,dt}. \quad \square$$

(b) Canonical form: $t^2 + 2t + 2 = (t+1)^2 + 1$. ☐

(c) **Sub:** $\boldsymbol{u = t + 1 \Rightarrow du = dt.}$ Bounds: $\boldsymbol{t = 0 \to u = 1;\ t = 1 \to u = 2.}$

Numerator: $t^2 + 1 = (u-1)^2 + 1 = u^2 - 2u + 2$. Denominator: $(t^2 + 2t + 2)^2 = ((u+1)^2 - 2u)^2$. More directly: $t^2 + 2t + 2 = (t+1)^2 + 1 = u^2 + 1$. So the denominator is $(u^2 + 1)^2$:

$$\Omega = \boxed{\frac{1}{2}\int_1^2 \frac{u^2 - 2u + 2}{(u^2+1)^2}\,du}. \quad \square$$

(d) Split $u^2 - 2u + 2 = (u^2 + 1) - 2u + 1$:

$$\Omega = \frac{1}{2}\int_1^2 \left[\frac{1}{u^2+1} - \frac{2u}{(u^2+1)^2} + \frac{1}{(u^2+1)^2}\right] du.$$

Standard results: $\displaystyle\int \frac{du}{u^2+1} = \arctan u$; $\displaystyle\int \frac{2u\,du}{(u^2+1)^2} = \frac{-1}{u^2+1}$; $\displaystyle\int \frac{du}{(u^2+1)^2} = \frac{u}{2(u^2+1)} + \frac{1}{2}\arctan u$.
Assembling:

$$\Omega = \frac{1}{2}\left[\arctan u + \frac{1}{u^2+1} + \frac{u}{2(u^2+1)} + \frac{1}{2}\arctan u\right]_1^2 = \frac{1}{2}\left[\frac{3}{2}\arctan u + \frac{u+2}{2(u^2+1)}\right]_1^2.$$

At $u = 2$: $\frac{3}{2}\arctan 2 + \frac{4}{10} = \frac{3}{2}\arctan 2 + \frac{2}{5}$. At $u = 1$: $\frac{3}{2}\cdot\frac{\pi}{4} + \frac{3}{4} = \frac{3\pi}{8} + \frac{3}{4}$.

$$\Omega = \frac{1}{2}\left(\frac{3}{2}\arctan 2 + \frac{2}{5} - \frac{3\pi}{8} - \frac{3}{4}\right) = \frac{3}{4}\arctan 2 - \frac{3\pi}{16} - \frac{7}{40}.$$

$$\boxed{\Omega = \frac{3}{4}\arctan 2 - \frac{3\pi}{16} - \frac{7}{40} \approx 0.0663}$$

**✓ Sanity Check:**

Numerical check: $\frac{3}{4}\arctan 2 \approx \frac{3}{4}(1.1071) \approx 0.8303$; $\frac{3\pi}{16} \approx 0.5890$; $\frac{7}{40} = 0.175$. Sum: $0.8303 - 0.5890 - 0.1750 \approx 0.0663$. This matches direct numerical integration of $\Omega$. ✓

**Bonus Part — Out of Program (For the Curious):**

**1.** $I_{34} = \displaystyle\int_{-\infty}^{+\infty} \frac{\sin x}{x}\, dx = \pi$ **(Dirichlet integral.)**

*Why it matters:* This is the cornerstone of Fourier analysis and signal processing. The Fourier transform of the rectangular pulse function (a perfect "gate" signal in electronics) is the sinc function $\frac{\sin x}{x}$. The value $\pi$ encodes the total energy of that pulse in frequency space. Engineers designing filters, communications systems, or MRI machines encounter this integral implicitly in every bandwidth calculation. It cannot be evaluated by elementary antiderivatives — it requires contour integration (complex analysis) or the Laplace transform.

**2.** $I_{35} = \displaystyle\int_0^1 x^{\lfloor 1/x \rfloor}\, dx$

*Why it matters:* This integral combines continuous integration with a discrete floor function, making it a bridge between analysis and number theory. It appears in the study of number-theoretic functions and Dirichlet series — the kind of mathematics that underlies modern cryptographic algorithms.

**3.** $I_{36} = \displaystyle\int_0^1 x^x\, dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^n} \approx 0.7834$ **(Sophomore's Dream.)**

*Why it matters:* The identity $\int_0^1 x^x\, dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^n}$ was discovered by Johann Bernoulli in 1697 and is a beautiful example of a closed-form series for a "non-elementary" integral. It connects power series, combinatorics, and analysis. In control theory and thermodynamics, integrals of the form $x^x$ appear in entropy calculations and partition functions, where the self-similar exponent structure has physical meaning.